Linear Algebra and its Applications 608 (2021) 270–281

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Complex unit gain graphs with exactly one positive eigenvalue

Lu Lu[a,*], Jianfeng Wang[b], Qiongxiang Huang[c]

[a] *School of Mathematics and Statistics, Central South University, Changsha, Hunan, 410083, China*
[b] *School of Mathematics and Statistics, Shandong University of Technology, Zibo 255049, China*
[c] *College of Mathematics and System Science, Xinjiang University, Urumqi, Xinjiang 830046, China*

A R T I C L E I N F O

Article history:
Received 27 July 2020
Accepted 11 September 2020
Available online 16 September 2020
Submitted by R. Brualdi

MSC:
05C50

Keywords:
Complex unit gain graphs
Positive eigenvalue
Spectrum

A B S T R A C T

A complex unit gain graph is a graph where each orientation of an edge is given a complex unit, which is the inverse of the complex unit assigned to the opposite orientation. In this paper, we characterize the structure of the complex unit gain graphs with exactly one positive eigenvalue. As its applications, we obtain the complex unit gain graphs with rank 2, and investigate the complex unit gain graphs with exactly two eigenvalues different from 0 and -1.

1. Introduction

Let G be a connected graph with vertex set V and edge set E, and let $\mathbb{T}$ be the multiplicative group of complex units, i.e., $\mathbb{T} = \{z \in \mathbb{C}^* \mid |z| = 1\}$. For any number $z \in \mathbb{C}$, denote by z^* the *conjugate* of z, and z_r and z_i the *real part* and the *imaginary*

* Corresponding author.
E-mail addresses: lulugdmath@163.com (L. Lu), jfwang@sdut.edu.cn (J. Wang), huangqxmath@163.com (Q. Huang).

https://doi.org/10.1016/j.laa.2020.09.016

part of z, respectively. The *complex unit gain graph* $\Phi = (G, \mathbb{T}, \varphi)$ is a graph with another structure φ: $\vec{E} \to \mathbb{T}$ such that $\varphi(vu) = \varphi(uv)^*$ for any $\{u, v\} \in E$, where $\vec{E} = \{uv, vu \mid \{u, v\} \in E\}$ is the set of oriented edges; see [13]. The graph G is the *underlying graph* of Φ and the function φ is the *gain function* of Φ. The *adjacency matrix* of Φ is an $n \times n$ matrix $A(\Phi) = [a_{ij}]$ defined by $a_{ij} = \varphi(v_i v_j)$ if $\{v_i, v_j\} \in E$ and 0 otherwise. Clearly, A is always a Hermitian matrix. Thus, the eigenvalues of A are real and can be ordered as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, where $n = |V|$. The eigenvalues of $A(\Phi)$ are called the *eigenvalues* of Φ. The multiset of all eigenvalues together with their multiplicities is the *spectrum* of Φ, denoted by $\mathrm{Sp}(\Phi)$. For convenience, we shall abbreviate the "complex unit gain graph" to *C-graph*.

Two C-graphs $\Phi = (G, \mathbb{T}, \varphi)$ and $\Phi' = (G, \mathbb{T}, \varphi')$ are *switching equivalent* if there is a map ζ: $V(G) \to \mathbb{T}$ such that $\varphi'(uv) = \zeta(u)^* \varphi(uv) \zeta(v)$. In this case, the map φ' can be written as $\varphi\zeta$. Note that $\mathrm{diag}(\zeta(v_1), \zeta(v_2), \ldots, \zeta(v_n))^{-1} = \mathrm{diag}(\zeta(v_1)^*, \zeta(v_2)^*, \ldots, \zeta(v_n)^*)$. It leads to that

$$A(\Phi') = \mathrm{diag}(\zeta(v_1), \zeta(v_2), \ldots, \zeta(v_n))^{-1} A(\Phi) \mathrm{diag}(\zeta(v_1), \zeta(v_2), \ldots, \zeta(v_n)).$$

Therefore, Φ and Φ' share the same spectrum. It is clear that the switching equivalence is an equivalent relation. Denote by $[\Phi]$ the equivalence class containing Φ. By simple observations, we have the following result.

Lemma 1. *Let $\Phi = (G, \mathbb{T}, \varphi)$ be a C-graph and $u \in V(G)$. Then there exists $\Phi' = (G, \mathbb{T}, \varphi') \in [\Phi]$ such that $\varphi'(uv) = 1$ for any $\{u, v\} \in E(G)$.*

Proof. Let ζ be such that $\zeta(v) = \varphi(vu)$ for any $v \in N(u)$ and $\zeta(x) = 1$ for any $x \in V(G) \setminus N(u)$. Therefore, $(\varphi\zeta)(vu) = \zeta(u)^* \varphi(uv) \zeta(v) = 1$ for any $u \sim v$. Thus, the C-graph $\Phi' = (G, \mathbb{T}, \varphi\zeta)$ is as desired. □

Clearly, each graph G can be regarded as a C-graph $\Phi = (G, \mathbb{T}, \varphi)$ with $\varphi(uv) = 1$ for any $\{u, v\} \in E(G)$. The adjacency matrix of this C-graph is just the adjacency matrix of the graph G. Recall that a *mixed graph* D is defined to be an ordered triple $(V, E, \mathcal{A})$, where V is the vertex set, and E and $\mathcal{A}$ are respectively the undirected edge set and the directed edge set with $E \cap \mathcal{A} = \emptyset$. The underlying graph of D is an undirected graph $\Gamma(D)$ with vertex set V and $u \sim v$ if either $uv \in \mathcal{A}$, $vu \in \mathcal{A}$ or $uv \in E$. The mixed graph D can be regarded as the C-graph $\Phi = (\Gamma(D), \mathbb{T}, \varphi)$ with $\varphi(uv) = 1$ if $uv \in E$, i if $uv \in \mathcal{A}$ or $-i$ if $vu \in \mathcal{A}$. Therefore, the adjacency matrix of the C-graph is just the Hermitian matrix of D, which is proposed by Liu and Li [10] and Guo and Mohar [8] independently. For the recent results on this topic, we turn the readers to see [1,4,9,12,14], for examples.

Throughout this paper, we always write K_n, $K_{n_1,n_2,\ldots,n_k}$ and P_n for the *complete graph*, the *complete multipartite graph* and the *path* respectively. For two positive numbers m, n, denote by e_n the *all-one vector* of length n, I_n the *identity matrix* of order n and $J_{m\times n}$ the *all-one matrix* of order $m \times n$. If the sizes of such matrices are clear, we delete the subscripts. Denote by $\overline{G}$ the *complement* of G and $G\nabla H$ the *join* of two graphs G and H. If $\varphi(uv) = 1$ for all $uv \in \vec{E}$, we simply denote $\Phi = (G, \mathbb{T}, \varphi)$ by G'.

We finally define a graph which will play an important role in the paper. Let $V_1, V_2, \dots, V_k$ be the color sets of $K_{n_1,n_2,\dots,n_k}$. If the C-graph $\Phi = (K_{n_1,n_2,\dots,n_k}, \mathbb{T}, \varphi)$ satisfies:

(a) $\varphi(v_s v_t) = \eta_{st}$ for any $v_s \in V_s$, $v_t \in V_t$ and $1 \le s < t \le k$;
(b) $\eta_{1t} = 1$ for any $2 \le t \le k$,

then Φ is called a *standard k-partite graph*, denoted by $(K_{n_1,n_2,\dots,n_k}; [\eta_{st} \mid 2 \le s < t \le k])$. Clearly, the undirected graph $K'_{n_1,n_2,\dots,n_k}$ is a standard k-partite graph, that is, $K'_{n_1,n_2,\dots,n_k} = (K_{n_1,n_2,\dots,n_k}; [1, 1, \dots, 1])$. In particular, we write $\vec{K}_{n_1,n_2,n_3}$ for $(K_{n_1,n_2,n_3}; [i])$.

The rest of the paper is organized as follows. In Section 2, some preliminary results are stated. In Section 3, we characterize the structure of C-graphs with exactly one positive eigenvalue, which urges us to determine the C-graphs with rank 2. In Section 4, we find a characterization of C-graphs with exactly one positive eigenvalue. As a derived result, we investigate the C-graphs with exactly two eigenvalues different from 0 and -1, and propose two problems for further study.

2. Preliminaries

Let $\Phi = (G, \mathbb{T}, \varphi)$ be a C-graph. For an induced subgraph H of G, the corresponding induced complex unit gain graph, denoted by $\Phi[H]$, is defined as $\Phi[H] = (H, \mathbb{T}, \varphi')$ satisfying $\varphi'(uv) = \varphi(uv)$ for every $uv \in \vec{E}$. If $V(H) = \{v_1, v_2, \dots, v_m\}$, then we also write $\Phi[v_1, v_2, \dots, v_m]$ for $\Phi[H]$. It is clear that the adjacency matrix $A(\Phi[H])$ is just the principal submatrix of $A(\Phi)$ induced by the vertex set of $V(H)$. Therefore, by Corollary 4.3.37 [5] we know that the famous interlacing theorem also holds.

Lemma 2. *Let $\Phi = (G, \mathbb{T}, \varphi)$ be a C-graph with order n, and H be a subgraph of G. If the eigenvalues of Φ and $\Phi[H]$ are $\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_n$ and $\mu_1 \ge \mu_2 \ge \dots \ge \mu_m$ respectively, then $\lambda_{n-m+i} \le \mu_i \le \lambda_i$ for $1 \le i \le m$.*

Let π: $V(G) = V_1 \cup V_2 \cup \dots \cup V_s$ be a partition of $V(G)$ with $|V_i| = n_i$ and $n = n_1 + n_2 + \dots + n_s$. For $1 \le j \le s$, each vertex set V_j is called a cell of the partition π. For $1 \le i, j \le s$, denote by $A_{i,j}$ the submatrix of $A(\Phi)$ whose rows are corresponding to V_i and columns are corresponding to V_j. Therefore, the adjacency matrix $A(\Phi)$ can be written as $A(\Phi) = [A_{ij}]$. Denote by $b_{ij} = e^T A_{ij} e / n_i$ the average row-sums of A_{ij}, where e denotes the all-one vector. The matrix $A_\pi = (b_{ij})_{s\times s}$ is called the *quotient matrix* of $A(\Phi)$. If, for any i, j, the row-sum of A_{ij} corresponding to any vertex $v \in V_i$ equals to b_{ij}, then π is called an *equitable partition* of Φ. Let δ_{V_i} be a vector indexed by $V(G)$ such that $\delta_{V_i}(v) = 1$ if $v \in V_i$ and 0 otherwise. The matrix $P = [\delta_{V_1} \delta_{V_2} \cdots \delta_{V_s}]$ is called the *characteristic matrix* of π. If π is an equitable partition, then $A(\Phi)P = PA_\pi$. It leads to the following famous result.

Lemma 3 *([7, Theorem 9.3.3, page 197]). Let Φ be a C-graph and π an equitable partition of Φ with quotient matrix A_π and characteristic matrix P. Then the eigenvalues of A_π are also eigenvalues of $A(\Phi)$. Furthermore, $A(\Phi)$ has the following two kinds of eigenvectors:*

(i) *the eigenvectors in the column space of P, and the corresponding eigenvalues coincide with the eigenvalues of A_π;*
(ii) *the eigenvectors orthogonal to the columns of P, i.e., those eigenvectors sum to zero on each cell of π.*

We end up this section by a characterization of complete multipartite graphs.

Lemma 4. *A connected graph is complete multipartite if and only if it contains no induced $K_2 \cup K_1$.*

Proof. Assume that G is a complete multi-partite graph. Suppose to the contrary that G contains induced $K_2 \cup K_1$, say $G[v_1, v_2, v_3] = K_2 \cup K_1$ and $G[v_1, v_2] = K_2$. Since $v_3 \nsim v_1$, the vertices v_1 and v_3 belong to a same color set. Since $v_3 \nsim v_2$, the vertices v_2 and v_3 belong to a same color set. Thus, v_1 and v_2 are in a same color set, which contradicts the fact $v_1 \sim v_2$. And the necessity follows.

Conversely, assume that the connected graph G contains no induced $K_2 \cup K_1$. Assume that G is k-chromatic graph and $U_1, U_2, \ldots, U_k$ are color sets of G. It is clear that there exists an edge between any two color sets. Suppose to the contrary that G is not complete multipartite. There exist two vertices in different color sets, say $u_1 \in U_1$ and $u_2 \in U_2$ such that $u_1 \nsim u_2$. Note that there is an edge between U_1 and U_2, say $u_1' \in U_1$ and $u_2' \in U_2$ such that $u_1' \sim u_2'$. If $u_1 \nsim u_2'$ then $G[u_1', u_2', u_1] = K_2 \cup K_1$; if $u_2 \nsim u_1'$ then $G[u_1', u_2', u_2] = K_2 \cup K_1$; if $u_1 \sim u_2'$ and $u_2 \sim u_1'$ then $G[u_1, u_2', u_2] = K_2 \cup K_1$. All cases lead to an induced $K_2 \cup K_1$, a contradiction. Hence, the sufficiency follows. $\square$

3. Main results

In this part, we first determine the C-graphs on n vertices with $\lambda_n \geq -1$. Secondly, we characterize the structure of C-graphs with exactly one positive eigenvalue. As an application, we obtain the C-graphs with rank 2, which is the main result in [15].

The fact that a connected graph is complete if and only if it contains no induced P_3 yields the following result.

Lemma 5. *If a connected C-graph $\Phi = (G, \mathbb{T}, \varphi)$ of order n has exactly one non-negative eigenvalue or $\lambda_n \geq -1$, then G is complete.*

Proof. Suppose to the contrary that P_3 is an induced subgraph of G. Since $\Phi[P_3]$ has spectrum $\{\sqrt{2}, 0, -\sqrt{2}\}$, we have $\lambda_2(\Phi) \geq 0$ and $\lambda_n(\Phi) \leq -\sqrt{2}$ by Lemma 2, a contradiction. $\square$

We now determine all C-graphs with $\lambda_n \geq -1$.

Proposition 1. *Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be all eigenvalues of the connected C-graph $\Phi = (G, \mathbb{T}, \varphi)$. Then $\lambda_n \geq -1$ if and only if $\Phi \in [K_n']$.*

Proof. The sufficiency is clear and we only prove the necessity. Since $\lambda_n \geq -1$, Lemma 5 means that G is complete. Set $V(G) = \{v_1, v_2, \ldots, v_n\}$. By Lemma 1, there exists $\Phi' = (G, \mathbb{T}, \varphi') \in [\Phi]$ such that $\varphi(v_1 v_j) = 1$ for $2 \leq j \leq n$. Now we consider $\Phi'[v_1, v_j, v_k]$ for any $\{j, k\} \subseteq \{2, 3, \ldots, n\}$. By Lemma 2, we have $\lambda_3(\Phi'[v_1, v_j, v_k]) \geq \lambda_n \geq -1$. Suppose that $\varphi'(v_j v_k) = a$. Then the characteristic polynomial of $\Phi'[v_1, v_j, v_k]$ is given by $f(x) = x^3 - 3x - 2a_r$. Note that $f(-1) = 2(1 - a_r)$. If $a_r < 1$, then $f(-1) > 0$, and thus $\lambda_3(\Phi'[v_1, v_j, v_k]) < -1$ due to the image of the function $f(x)$, which is a contradiction. Thus, we have $a_r = 1$, that is $a = 1$. It leads to $\varphi'(v_j v_k) = 1$ for any j, k. Thus, $\Phi' = K_n'$. $\square$

We next consider the C-graphs with $\lambda_2 \leq 0$.

Lemma 6. *If the connected C-graph $\Phi = (G, \mathbb{T}, \varphi)$ of order n has exactly one positive eigenvalue then G is complete multipartite.*

Proof. Sine the connected graphs with $2 \leq n \leq 3$ are complete multipartite, we may assume $n \geq 4$. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of Φ with $\lambda_1 > 0$ and $\lambda_2 \leq 0$. According to Lemma 4, it suffices to show that G contains no induced $K_2 \cup K_1$. Suppose to the contrary that $K_2 \cup K_1$ is an induced subgraph of G. It is clear that one of $2K_2$, P_4 and $K_{1,3}^+$ is an induced subgraph of G, where $K_{1,3}^+$ is the graph obtained from $K_{1,3}$ by adding a new edge. Denote by $V(K_{1,3}^+) = \{v_1, v_2, v_3, v_4\}$ such that the degrees $d(v_1) = 1$ and $d(v_2) = 3$. Assume without loss of generality that $\varphi(v_2 v_1) = \varphi(v_2 v_3) = \varphi(v_2 v_4) = 1$ and $\varphi(v_3 v_4) = a$. By immediate calculations, the characteristic polynomial of $\Phi[K_{1,3}^+]$ is $f(x) = x^4 - 4x^2 - 2a_r x + 1$. Since $f(0) = 1 > 0$ and $f(x)$ has a positive root, by the image of the function $f(x)$, we have $\lambda_2(K_{1,3}^+) > 0$. Note that we also have $\lambda_2(\Phi[2K_2]) = 1 > 0$ and $\lambda_2(\Phi[P_4]) \approx 0.62 > 0$. By Lemma 2, we have $\lambda_2 > 0$, a contradiction. $\square$

Lemma 7. *Let $\Phi = (G, \mathbb{T}, \varphi)$ be a C-graph with $G = K_{n_1, n_2, \ldots, n_k}$ and $V_1, V_2, \ldots, V_k$ being the color sets of G. If $\lambda_2 \leq 0$, then there exists a standard k-partite graph $\Phi' = (K_{n_1, n_2, \ldots, n_k}; [\eta_{st} \mid 2 \leq s < t \leq k])$ such that $\Phi \in [\Phi']$.*

Proof. By Lemma 2, any induced C-graph $\Phi[H]$ of Φ satisfies $\lambda_2(\Phi[H]) \leq \lambda_2(\Phi) = 0$. This fact will be used frequently. Assume that $V_j = \{v_1^{(j)}, v_2^{(j)}, \ldots, v_{n_j}^{(j)}\}$ for $1 \leq j \leq k$. Lemma 1 implies that there exists $\Phi_1 = (G, \mathbb{T}, \varphi_1) \in [\Phi]$ such that $\varphi_1(v_1^{(1)} v) = 1$ for any

$v \in V(G) \setminus V_1$. For any $2 \le s \le k$ and $v_a^{(s)}, v_b^{(s)} \in V_s$, the characteristic polynomial of $\Phi_1[v_1^{(1)}, v_2^{(1)}, v_a^{(s)}, v_b^{(s)}]$ is

$$f(x) = \det \begin{bmatrix} x & 0 & -1 & -1 \\ 0 & x & -\varphi_1(v_2^{(1)} v_a^{(s)}) & -\varphi_1(v_2^{(1)} v_b^{(s)}) \\ -1 & -\varphi_1(v_2^{(1)} v_a^{(s)})^* & x & 0 \\ -1 & -\varphi_1(v_2^{(1)} v_b^{(s)})^* & 0 & x \end{bmatrix}$$

$$= x^4 - 4x^2 + 2(1 - (\varphi_1(v_2^{(1)} v_a^{(s)})\varphi_1(v_2^{(1)} v_b^{(s)})^*)_r).$$

Since $\lambda_2(\Phi_1[v_1^{(1)}, v_2^{(1)}, v_a^{(s)}, v_b^{(s)}]) \le 0$, we have $f(0) = 2(1-(\varphi_1(v_2^{(1)} v_a^{(s)})\varphi_1(v_2^{(1)} v_b^{(s)})^*)_r) \le 0$ which leads to $\varphi_1(v_2^{(1)} v_a^{(s)})\varphi_1(v_2^{(1)} v_b^{(s)})^* = 1$. Thus, $\varphi_1(v_2^{(1)} v_a^{(s)}) = \varphi_1(v_2^{(1)} v_b^{(s)})$. Thereby, we have $\varphi_1(v_2^{(1)} v_a^{(s)}) = c_{1s}$ is a constant only dependent on s. By taking a map ζ such that $\zeta(v_2^{(1)}) = c_{12}$ and $\zeta(x) = 1$ for any other vertex x we obtain $(\varphi_1\zeta)(v_2^{(1)} v_a^{(2)}) = 1$ for any $v_a^{(2)} \in V_2$ and $(\varphi_1\zeta)(uv) = \varphi_1(uv)$ for any $u \sim v$ and $u, v \ne v_2^{(1)}$. Thus, by taking $\varphi_2 = (\varphi_1\zeta)$ and $\Phi_2 = (G, \mathbb{T}, \varphi_2)$, we have $\varphi_2(v_1^{(1)} v) = 1$ for any $v \in V(G) \setminus V_1$ and $\varphi_2(v_2^{(1)} v_a^{(s)}) = c_{12}^* c_{1s} = c'_{1s}$ for any $v_a^{(s)} \in V_s$. In particular, $c'_{12} = 1$. Note that for any $3 \le t \le k$ and $v_b^{(t)} \in V_t$, the characteristic polynomial of $\Phi_2[v_1^{(1)}, v_2^{(1)}, v_1^{(2)}, v_b^{(t)}]$ is

$$g(x) = \det \begin{bmatrix} x & 0 & -1 & -1 \\ 0 & x & -1 & -c'_{1t} \\ -1 & -1 & x & -\varphi_2(v_1^{(2)} v_b^{(t)}) \\ -1 & -(c'_{1t})^* & -\varphi_2(v_1^{(2)} v_b^{(t)})^* & x \end{bmatrix}$$

$$= x^4 - 5x^2 - 2(\varphi_2(v_1^{(2)} v_b^{(t)}) + c'_{1t}\varphi_2(v_1^{(2)} v_b^{(t)})^*)_r x + 2(1 - (c'_{1t})_r).$$

Since $\lambda_2(\Phi_2[v_1^{(1)}, v_2^{(1)}, v_1^{(2)}, v_b^{(t)}]) \le 0$, we have $g(0) = 2(1 - (c'_{1t})_r) \le 0$. It leads to $c'_{1t} = 1$ for any $3 \le t \le k$. Therefore, we have $\varphi_2(v_1^{(1)} v) = \varphi_2(v_2^{(1)} v) = 1$ for any $v \in V(G) \setminus V_1$. Similarly, by considering $v_j^{(1)}$ for every $3 \le j \le n_1$, we can ultimately obtain $\Phi' = (G, \mathbb{T}, \varphi')$ such that $\varphi'(v_x^{(1)} v) = 1$ for any $v_x^{(1)} \in V_1$ and $v \in V(G) \setminus V_1$.

In what follows, we show that $\varphi'(v_a^{(s)} v_b^{(t)})$ is a complex number η_{st} not really depending on a and b. Suppose to the contrary that there exist $\varphi'(v_a^{(s)} v_b^{(t)}) \ne \varphi'(v_{a'}^{(s)} v_{b'}^{(t)})$ for some $2 \le s, t \le k$. Note that the characteristic polynomial of $\Phi'[v_1^{(1)}, v_a^{(s)}, v_b^{(t)}, v_{b'}^{(t)}]$ is

$$f_1(x) = \det \begin{bmatrix} x & -1 & -1 & -1 \\ -1 & x & -\varphi'(v_a^{(s)} v_b^{(t)}) & -\varphi'(v_a^{(s)} v_{b'}^{(t)}) \\ -1 & -\varphi'(v_a^{(s)} v_b^{(t)})^* & x & 0 \\ -1 & \varphi'(v_a^{(s)} v_{b'}^{(t)})^* & 0 & x \end{bmatrix}$$

$$= x^4 - 5x^2 + 2(1 - (\varphi'(v_a^{(s)} v_b^{(t)})\varphi'(v_a^{(s)} v_{b'}^{(t)})^*)_r),$$

and the characteristic polynomial of $\Phi'[v_1^{(1)}, v_{b'}^{(t)}, v_a^{(s)}, v_{a'}^{(s)}]$ is

$$\begin{aligned} f_2(x) &= \det \begin{bmatrix} x & -1 & -1 & -1 \\ -1 & x & -\varphi'(v_{b'}^{(t)} v_a^{(s)}) & -\varphi'(v_{b'}^{(t)} v_{a'}^{(s)}) \\ -1 & -\varphi'(v_{b'}^{(t)} v_a^{(s)})^* & x & 0 \\ -1 & -\varphi'(v_{b'}^{(t)} v_{a'}^{(s)})^* & 0 & x \end{bmatrix} \\ &= x^4 - 5x^2 + 2(1 - (\varphi'(v_{b'}^{(t)} v_a^{(s)})\varphi'(v_{b'}^{(t)} v_{a'}^{(s)})^*)_r). \end{aligned}$$

Since $\lambda_2(\Phi'[v_1^{(1)}, v_a^{(s)}, v_b^{(t)}, v_{b'}^{(t)}]), \lambda_2(\Phi'[v_1^{(1)}, v_{b'}^{(t)}, v_a^{(s)}, v_{a'}^{(s)}]) \leq 0$, we get

$$\begin{aligned} f_1(0) &= 2(1 - (\varphi'(v_a^{(s)} v_b^{(t)})\varphi'(v_a^{(s)} v_{b'}^{(t)})^*)_r) \leq 0 \quad \text{and} \\ f_2(0) &= 2(1 - (\varphi'(v_{b'}^{(t)} v_a^{(s)})\varphi'(v_{b'}^{(t)} v_{a'}^{(s)})^*)_r) \leq 0, \end{aligned}$$

which result in $\varphi'(v_a^{(s)} v_b^{(t)}) = \varphi'(v_a^{(s)} v_{b'}^{(t)})$ and $\varphi'(v_{b'}^{(t)} v_a^{(s)}) = \varphi'(v_{b'}^{(t)} v_{a'}^{(s)})$. Therefore, $\varphi'(v_a^{(s)} v_b^{(t)}) = \varphi'(v_{a'}^{(s)} v_{b'}^{(t)})$, a contradiction.

The proof is completed. □

The following result follows from combining Lemmas 6 and 7.

Theorem 1. *If the connected C-graph $\Phi = (G, \mathbb{T}, \varphi)$ has exactly one positive eigenvalue, then there exists a standard k-partite graph $\Phi' = (K_{n_1,n_2,\ldots,n_k}; [\eta_{st} \mid 2 \leq s < t \leq k])$ such that $\Phi \in [\Phi']$.*

As an application of the above theorem, we can characterize the C-graphs with rank 2, which is the main result in the paper [15].

Proposition 2. *Let $\Phi = (G, \mathbb{T}, \varphi)$ be a connected C-graph of order n. Then Φ has rank 2 if and only if $\Phi \in [K'_{n_1,n_2}]$ for some positive integers n_1, n_2 with $n_1 + n_2 = n$ or $\Phi \in [\vec{K}_{l_1,l_2,l_3}]$ for some positive integers l_1, l_2, l_3 with $l_1 + l_2 + l_3 = n$.*

Proof. Assume that $\Phi = (G, \mathbb{T}, \varphi)$ has spectrum $\{\alpha, [0]^{n-2}, \beta\}$. By Theorem 1, $G \in [\Phi']$ where $\Phi' = [(K_{n_1,n_2,\ldots,n_k}; [\eta_{st} \mid 2 \leq s < t \leq k])]$. Since any principal minor of $A(\Phi')$ with order 3 has determinant 0, for any $1 \leq l, s, t \leq k$, we have

$$\det A(\Phi[v_1^{(l)}, v_1^{(s)}, v_1^{(t)}]) = \det \begin{bmatrix} 0 & \eta_{ls} & \eta_{lt} \\ \eta_{sl} & 0 & \eta_{st} \\ \eta_{tl} & \eta_{ts} & 0 \end{bmatrix} = 2(\eta_{ls}\eta_{st}\eta_{tl})_r = 0.$$

It leads to that $\eta_{ls}\eta_{st}\eta_{tl} = \pm i$. Suppose that $k \geq 4$. Therefore, by taking $\{l, s, t\} = \{1, 2, 3\}$, $\{1, 2, 4\}$ and $\{1, 3, 4\}$ respectively, we have

$$\begin{cases} \eta_{12}\eta_{23}\eta_{31} = \eta_{23} = \pm i, \\ \eta_{12}\eta_{24}\eta_{41} = \eta_{24} = \pm i, \\ \eta_{13}\eta_{34}\eta_{41} = \eta_{34} = \pm i. \end{cases}$$

Therefore, we have

$$\begin{aligned} \det A(\Phi[v_1^{(1)}v_1^{(2)}v_1^{(3)}v_1^{(4)}]) &= \det \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & \eta_{23} & \eta_{24} \\ 1 & \eta_{23}^* & 0 & \eta_{34} \\ 1 & \eta_{24}^* & \eta_{34}^* & 0 \end{bmatrix} \\ &= 3 - 2(\eta_{23}\eta_{34} + \eta_{23}\eta_{24}^* + \eta_{24}\eta_{34}^*)_r \neq 0, \end{aligned}$$

which is a contradiction. Thus, we have $k \leq 3$. If $k = 2$, then $\Phi' = K_{n_1,n_2}$ by $\eta_{12} = 1$. If $k = 3$, then $\Phi' = \vec{K}_{l_1,l_2,l_3}$ by $\eta_{23} = \pm i$.

Conversely, it is well-known that K_{n_1,n_2} has only one positive eigenvalue. In fact, $\mathrm{Sp}(K_{n_1,n_2}) = \{\sqrt{n_1n_2}, [0]^{n-2}, -\sqrt{n_1n_2}\}$. Assume that the color sets of K_{n_1,n_2,n_3} are V_1, V_2, V_3 and $|V_j| = n_j$ for $1 \leq j \leq 3$. Denote by $\delta_{v_s - v_t}$ the vector with $\delta_{v_s-v_t}(x) = 1$ if $x = v_s$, -1 if $x = v_t$, and 0 otherwise. It is easy to verify that $A(\Phi)\delta_{v_s-v_t} = 0$ for any $\{v_s, v_t\} \subseteq V_j$ and $1 \leq j \leq 3$. It implies that 0 is an eigenvalue of Φ with multiplicity at least $n - 3$. Note that π: $V = V_1 \cup V_2 \cup V_3$ is an equitable partition of Φ with quotient matrix

$$A_\pi = \begin{pmatrix} 0 & n_2 & n_3 \\ n_1 & 0 & n_3 i \\ n_1 & -n_2 i & 0 \end{pmatrix},$$

whose eigenvalues are 0 and $\pm\sqrt{n_1n_2 + n_2n_3 + n_3n_1}$. Thus, Lemma 3 indicates that $\mathrm{Sp}(\vec{K}_{n_1,n_2,n_3}) = \{\sqrt{n_1n_2 + n_2n_3 + n_3n_1}, [0]^{n-2}, -\sqrt{n_1n_2 + n_2n_3 + n_3n_1}\}$, which has exactly one positive eigenvalue.

This completes the proof. $\square$

Note that the converse of Theorem 1 is not true. For example, the standard 3-partite graph $\Phi = (K_{1,1,1}; [-1])$ has spectrum $\{[1]^2, -2\}$. In the next section, we will further investigate the conditions for a C-graph having exactly one positive eigenvalue.

4. Further results and discussions

Let $\Phi = (K_{n_1,n_2,\ldots,n_k}; [\eta_{st} \mid 2 \leq s < t \leq k])$ be a standard k-partite graph with color sets $V_1, V_2, \ldots, V_k$. It is clear that π: $V = V_1 \cup V_2 \cup \cdots \cup V_k$ is an equitable partition of Φ with quotient matrix

$$A_\pi = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & \eta_{23} & \cdots & \eta_{2k} \\ 1 & \eta_{32} & 0 & \cdots & \eta_{3k} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \eta_{k2} & \eta_{k3} & \cdots & 0 \end{bmatrix} \cdot \begin{bmatrix} n_1 & 0 & 0 & \cdots & 0 \\ 0 & n_2 & 0 & \cdots & 0 \\ 0 & 0 & n_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & n_k \end{bmatrix}.$$

As similar to the proof of the sufficiency part of Corollary 2, the spectrum of Φ consists of 0 with multiplicity $n-k$ and the k eigenvalues of A_π, due to Lemma 3. Thus, Φ has exactly one positive eigenvalue if and only if A_π has exactly one positive eigenvalue. Note that the matrix A_π is congruent to the matrix

$$A' = \begin{bmatrix} n_1 & 0 & 0 & \cdots & 0 \\ 0 & n_2 & 0 & \cdots & 0 \\ 0 & 0 & n_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & n_k \end{bmatrix}^{-1/2} \cdot \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & \eta_{23} & \cdots & \eta_{2k} \\ 1 & \eta_{32} & 0 & \cdots & \eta_{3k} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \eta_{k2} & \eta_{k3} & \cdots & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} n_1 & 0 & 0 & \cdots & 0 \\ 0 & n_2 & 0 & \cdots & 0 \\ 0 & 0 & n_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & n_k \end{bmatrix}^{1/2},$$

which is similar to the matrix

$$A'' = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & \eta_{23} & \cdots & \eta_{2k} \\ 1 & \eta_{32} & 0 & \cdots & \eta_{3k} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \eta_{k2} & \eta_{k3} & \cdots & 0 \end{bmatrix}.$$

Therefore, A_π has exactly one positive eigenvalue if and only if A'' has exactly one positive eigenvalue. Let $\Phi_\pi = (K_k, \mathbb{T}, \varphi)$ be a C-graph such that $V(K_n) = \{v_1, v_2, \ldots, v_k\}$ and $\varphi(v_1 v_s) = 1$ for any $2 \le s \le k$ and $\varphi(v_s v_t) = \eta_{st}$ for $2 \le s < t \le k$. The C-graph Φ_π is called the *congruent quotient graph* of the standard k-partite graph Φ. Clearly, the adjacency matrix of Φ_π is A''. Thus, we have shown the following result.

Lemma 8. *The standard k-partite graph $\Phi = (K_{n_1,n_2,\ldots,n_k}; [\eta_{st} \mid 2 \le s < t \le k])$ has exactly one positive eigenvalue if and only if Φ_π has exactly one positive eigenvalue.*

Combining Theorem 1 and Lemma 8, we get the following result.

Theorem 2. *The connected C-graph $\Phi = (G, \mathbb{T}, \varphi)$ has exactly one positive eigenvalue if and only if there exists a standard k-partite graph $\Phi' = (K_{n_1,n_2,\ldots,n_k}; [\eta_{st} \mid 2 \leq s < t \leq k])$, whose congruent quotient graph Φ'_π has exactly one positive eigenvalue, such that $\Phi \in [\Phi']$.*

As stated above, to completely determine the C-graphs with exactly one positive eigenvalue, it only needs to determine the Φ_π with exactly one positive eigenvalue.

Lemma 9. *Let $\Phi = (K_n, \mathbb{T}, \varphi)$ be a connected C-graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and $\varphi(v_1 v_s) = 1$ for any $2 \leq s \leq n$. If Φ has exactly one positive eigenvalue, then $(\varphi(v_s v_t))_r \geq 0$ for any $2 \leq s < t \leq n$ and $(\varphi(v_a v_b)\varphi(v_b v_c)\varphi(v_c v_a))_r \geq 0$ for any $2 \leq a < b < c \leq n$.*

Proof. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of Φ. Since $\lambda_2 \leq 0$, for any induced subgraph H of K_n we get $\lambda_2(\Phi[H]) \leq 0$ by Lemma 2. For any $2 \leq s < t \leq n$, the characteristic polynomial of $\Phi[v_1, v_s, v_t]$ is $f(x) = x^3 - 3x - 2(\varphi(v_s v_t))_r$. Since $\lambda_2(\Phi[v_1, v_s, v_t]) \leq 0$, we have $f(0) = -2(\varphi(v_s v_t))_r \leq 0$, and thus $\varphi(v_s v_t)_r \geq 0$. By considering $\Phi[v_a, v_b, v_c]$, one can similarly obtain $(\varphi(v_a v_b)\varphi(v_b v_c)\varphi(v_c v_a))_r \geq 0$. $\square$

Unfortunately, it could happen that a C-graph $\Phi = (K_n, \mathbb{T}, \varphi)$ with $V = \{v_1, v_2, \ldots, v_n\}$ satisfies

$$(\varphi(v_s v_t))_r \geq 0 \text{ for } 2 \leq s < t \leq n, \text{ and } (\varphi(v_a v_b)\varphi(v_b v_c)\varphi(v_c v_a))_r \geq 0$$
$$\text{for } 2 \leq a < b < c \leq n,$$

yet Φ has more than one positive eigenvalue. For example, the C-graph $\Phi = (K_4, \mathbb{T}, \varphi)$ with $\varphi(v_1 v_s) = 1$ for $2 \leq s \leq 4$, $\varphi(v_2 v_3) = 1/\sqrt{2} - i/\sqrt{2}$, $\varphi(v_3 v_4) = 1/\sqrt{2} + i/\sqrt{2}$ and $\varphi(v_4 v_2) = 7/25 + 24i/25$ has spectrum $\{2.72, 0.05, -0.78, -1.99\}$. It satisfies the conditions in Lemma 9 but has two positive eigenvalues. Remark, Lemma 1 implies that any C-graph with underlying graph being the complete graph K_k corresponds to a graph Φ_π. We propose the following problem to end this topic.

Problem 1. *Completely determine the C-graphs $\Phi = (K_n, \mathbb{T}, \varphi)$ with exactly one positive eigenvalue.*

Recently, research on graphs with two eigenvalues having high multiplicity has attracted attention. With respect to adjacency matrix of undirected graphs, Cioabă, Haemers, and Vermette [2,3] characterized the graphs with all but two eigenvalues equals to ± 1, and the graphs with all but two eigenvalues equal to -2 or 0. For the distance matrix of undirected graphs, Lu et al. [11] characterized the graphs with all but two distance eigenvalues equal to -1 or -3. Huang et al. [6] determined the graphs with all but at most three distance eigenvalues equal to 0 or -2. Motivated by such works, we try to characterize the C-graph $\Phi = (G, \mathbb{T}, \varphi)$ with exactly two eigenvalues different from 0 and -1. By

Corollary 1, such Φ has exactly one positive eigenvalue and one negative eigenvalue less than -1. Therefore, by Theorem 1, there exists $\Phi' = (K_{n_1,n_2,\ldots,n_k}; [\eta_{st} \mid 2 \le s < t \le k])$ such that $\Phi \in [\Phi']$.

By arguments above, to make sure that Φ' has exactly two eigenvalues different from 0 and -1, it only needs to make sure that A_π has exactly two eigenvalues different from 0 and -1. However, it is not easy to deal with this problem. We here present a result below.

Theorem 3. *Let $\Phi = (G, \mathbb{T}, \varphi)$ be a connected C-graph of order n such that G contains no K_4 as its subgraph. Then Φ has exactly two eigenvalues different from* 0 *and* -1 *if and only if $\Phi \in [\Phi']$, where Φ' is K_{n_1,n_2}, $\vec{K}_{n_1,n_2,n_3}$ or $(K_{n_1,n_2,n_3}; [\eta_{23}])$ with $(\eta_{23})_r = \frac{n_1n_2+n_2n_3+n_3n_1-1}{2n_1n_2n_3}$.*

Proof. The sufficiency follows from direct calculations. We next show the necessity. Suppose that -1 is not an eigenvalue of Φ. Then, Φ has rank 2, and Corollary 2 leads to $\Phi \in [K_{n_1,n_2}]$ or $\Phi \in [\vec{K}_{n_1,n_2,n_3}]$. Suppose that -1 is an eigenvalue of Φ. Since Φ has only one positive eigenvalue, Theorem 1 indicates that there exists $\Phi' = (K_{n_1,n_2,\ldots,n_k}; [\eta_{st} \mid 2 \le s < t \le k])$ such that $\Phi \in [\Phi']$. Since G contains no K_4, we have $k \in \{2, 3\}$. If $k = 2$ then $\Phi' = K_{n_1,n_2}$ which has no -1 as an eigenvalue. If $k = 3$ then $\Phi' = (K_{n_1,n_2,n_3}; [\eta_{23}])$ and

$$A_\pi = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & \eta_{23} \\ 1 & \eta_{32} & 0 \end{bmatrix} \cdot \begin{bmatrix} n_1 & 0 & 0 \\ 0 & n_2 & 0 \\ 0 & 0 & n_3 \end{bmatrix}.$$

Therefore, Φ' has -1 as an eigenvalue if and only if A_π has -1 as an eigenvalue if and only if $\det(A_\pi + I) = 0$ if and only if $(\eta_{23})_r = \frac{n_1n_2+n_2n_3+n_3n_1-1}{2n_1n_2n_3}$.

The proof is completed. $\square$

We end up this paper by proposing the following problem.

Problem 2. *Completely determine the C-graphs with exactly two eigenvalues different from* 0 *and* -1.

Declaration of competing interest

There is no conflict of interest.

Acknowledgements

The second author and the third author are supported for this research by the National Natural Science Foundation of China (Nos. 11971274, 11671344). The authors are very grateful to the editors and the referees for their valuable comments and corrections.

References

[1] F. Belardo, M. Brunetti, M. Cavaleri, A. Donnob, Godsil-McKay switching for mixed and gain graphs over the circle group, Linear Algebra Appl., https://doi.org/10.1016/j.laa.2020.04.025.

[2] S. Cioabă, W. Haemers, J. Vermette, W. Wong, The graphs with all but two eigenvalues equal to ± 1, J. Algebraic Comb. 41 (2015) 887–897.

[3] S. Cioabă, W. Haemers, J. Vermette, The graphs with all but two eigenvalues equal to -2 or 0, Des. Codes Cryptogr. 84 (2016) 153–163.

[4] S.J. He, R.-X. Hao, F.M. Dong, The rank of a complex unit gain graph in terms of the matching number, Linear Algebra Appl. 589 (2020) 158–185.

[5] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, 2013.

[6] X. Huang, Q. Huang, L. Lu, Graphs with at most three distance eigenvalues different from -1 and -2, Graphs Comb. 34 (2018) 395–414.

[7] C.D. Godsil, G. Royle, Algebraic Graph Theory, Springer-Verlag, Berlin, 2001.

[8] K. Guo, B. Mohar, Hermitian adjacency matrix of digraphs and mixed graphs, J. Graph Theory 85 (2016) 217–248.

[9] S. Li, W. Wei, The multiplicity of an A-eigenvalue: a unified approach for mixed graphs and complex unit gain graphs, Discrete Math. 343 (2020) 111917.

[10] J. Liu, X. Li, Hermitian-adjacency matrix and Hermitian energies of mixed graphs, Linear Algebra Appl. 466 (2015) 182–207.

[11] L. Lu, Q. Huang, X. Huang, The graphs with exactly two distance eigenvalues different from -1 and -3, J. Algebraic Comb. 45 (2017) 629–647.

[12] Y. Lu, L.G. Wang, P. Xiao, Complex unit gain bicyclic graphs with rank 2, 3 or 4, Linear Algebra Appl. 523 (2017) 169–186.

[13] N. Reff, Spectral properties of complex unit gain graphs, Linear Algebra Appl. 436 (2012) 3165–3176.

[14] Y. Wang, S.-C. Gong, Y.-Z. Fan, On the determinant of the Laplacian matrix of a complex unit gain graph, Discrete Math. 341 (2018) 81–86.

[15] F. Xu, Q. Zhou, D. Wong, F.L. Tian, Complex unit gain graphs of rank 2, Linear Algebra Appl. 597 (2020) 155–169.