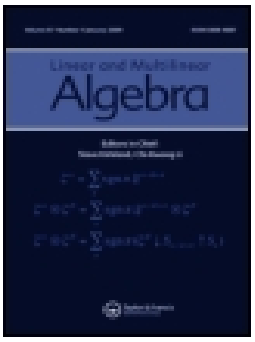

Linear and Multilinear Algebra

ISSN: 0308-1087 (Print) 1563-5139 (Online) Journal homepage: https://www.tandfonline.com/loi/glma20

Distance eigenvalues of $B(n, k)$

Lu Lu & Qiongxiang Huang

LINEAR AND MULTILINEAR ALGEBRA
https://doi.org/10.1080/03081087.2019.1659221

Distance eigenvalues of $B(n,k)$

Lu Lu[a] and Qiongxiang Huang[b]

[a]School of Mathematics and Statistics, Central South University, Changsha, People's Republic of China;
[b]College of Mathematics and Systems Science, Xinjiang University, Urumqi, People's Republic of China

ABSTRACT

Let $B(n,k)$ be the subgraph of the Boolean lattice BL_n induced by the kth and $(k+1)$th layers. The distance spectrum of BL_n was obtained by Aalipour et al. and Koolen et al. proved that BL_n is determined by its distance spectrum. In this paper, we introduce a new method to verify that a quotient matrix contains all distance eigenvalues of a graph, and we get the distance spectrum of $B(n,k)$ by using this method. It is proved that $B(n,k)$ has exactly four distinct distance eigenvalues, and its diameter can be arbitrarily large.

ARTICLE HISTORY
Received 7 January 2019
Accepted 26 July 2019

COMMUNICATED BY
S. Cioaba

KEYWORDS
Distance spectrum; equitable partition; Boolean lattice

AMS SUBJECT CLASSIFICATIONS
05C50; 05C12

1. Introduction

In this paper, we only consider connected simple graphs. Let $\Gamma = (V, E)$ be a connected graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{e_1, e_2, \ldots, e_m\}$. The *distance* between v_i and v_j, denoted by $d(v_i, v_j)$ (or $d_{i,j}$ for short), is defined as the length of a shortest path between v_i and v_j. The *diameter* of Γ, denoted by $d(\Gamma)$, is the largest distance in Γ. The *distance matrix* of Γ, denoted by $\mathcal{D}(\Gamma)$, is the $n \times n$ matrix whose (i,j)-entry is equal to $d_{i,j}$, for $1 \le i, j \le n$. The *distance eigenvalues* of Γ are the eigenvalues of $\mathcal{D}(\Gamma)$. Let $\partial_1 \ge \partial_2 \ge \cdots \ge \partial_s$ be all distinct eigenvalues of $\mathcal{D}(\Gamma)$ with multiplicities $m_1, m_2, \ldots, m_s$. By the Perron–Frobenius theorem (see [1, Theorem 2.2.1]), we have $\partial_1 \ge |\partial_t|$ and $m_1 = 1$. The multiset of such eigenvalues together with their multiplicities is the *distance spectrum* of Γ, denoted by $\mathrm{Sp}_{\mathcal{D}}(\Gamma) = \{\partial_1, \partial_2, \ldots, \partial_n\}$. For more details about distance eigenvalues, we refer the reader to [2–4].

The *Boolean lattice* BL_n (or *hypercube* Q_n) is the graph whose vertices are all subsets of $[n] = \{1, \ldots, n\}$ and two subsets are adjacent if their symmetric difference has precisely one element. The *kth layer* L_k of BL_n is the family of all k-subsets of $[n]$. For $0 \le k \le n-1$, let $B(n,k)$ denote the subgraph of BL_n induced by the layers L_k and L_{k+1}. In detail, the graph $B(n,k)$ has vertex set $V(B(n,k)) = \{S \subseteq [n] \mid |S| = k \text{ or } k+1\}$ and two vertices S_1 and S_2 are adjacent if $S_1 \subset S_2$ or $S_2 \subset S_1$. The distance spectrum of BL_n was given by Aalipour et al. [2], and Koolen et al. [5] proved that BL_n is determined by its distance spectrum, that is, there is no graph sharing the same distance spectrum with BL_n but not isomorphic to

CONTACT Qiongxiang Huang huangqx@xju.edu.cn, huangqxmath@163.com College of Mathematics and Systems Science, Xinjiang University, Urumqi, Xinjiang 830046, People's Republic of China

it. Recently, Mirafzal [6] obtained the adjacency eigenvalues of the line graph of $B(n,1)$, and this result was generalized by Huang and Huang [7], who obtained the adjacency eigenvalues of the line graph of $B(n,k)$ for $1 \leq k \leq n-1$.

The elementary problem to investigate the distance spectrum of a graph is to calculate its distance spectrum. However, there are no general strategies to calculate the distance spectrum of a graph up to now. One may obtain some distance eigenvalues of a graph by using the knowledge of equitable partition and quotient matrix, which will be introduced in Section 2. Moreover, Godsil and Royle [8, Theorem 9.4.1] give a condition that any eigenvalue of a graph is also an eigenvalue of a quotient matrix. Motivated by this idea, we obtain a quotient matrix of the distance matrix of $B(n,k)$ which contains all distance eigenvalues of $B(n,k)$ by using a totally different method. We think such a method may be useful to obtain the distance spectra of some other graphs.

It is known that any connected graph with diameter d has at least $d+1$ distinct adjacency eigenvalues [9]. However, it is not true for distance eigenvalues. In [10], the authors proposed the question 'Are there connected graphs that are not distance regular with diameter d and having less than $d+1$ distance eigenvalues?' This question was answered by Aalipour et al. [2], who constructed a class of non-regular connected graphs having at most 5 distinct distance eigenvalues with diameter $d+1$ for any d. In this paper, we show that $B(n,k)$ has exactly 4 distinct distance eigenvalues with diameter not less than $2k+1$ by completely determining the distance spectrum of it.

2. Preliminaries

In this part, we introduce the knowledge of the equitable partition of a symmetric real matrix. The knowledge of equitable partition contains very rich content, and it is a very powerful tool in spectral graph theory. Here we give a brief description of it and refer the reader to [1, Section 2.3] or [8, Section 9.3] for details.

Suppose that M is an $n \times n$ symmetric real matrix whose rows and columns are indexed by $X = \{1, \dots, n\}$. Let Π be a partition of X with cells $X_1, \dots, X_m$. The matrix M can be written as

$$M = \begin{pmatrix} M_{1,1} & \cdots & M_{1,m} \\ \vdots & & \vdots \\ M_{m,1} & \cdots & M_{m,m} \end{pmatrix}$$

where $M_{i,j}$ is the submatrix of M whose rows and columns are induced by X_i and X_j, respectively, for $1 \leq i,j \leq m$. Let b_{ij} be the average row sum of $M_{i,j}$. Then $B_m = (b_{ij})_{m\times m}$ is the *quotient matrix* of M with respect to the partition Π. Especially, if the row sum of each block $M_{i,j}$ is a constant, then the partition is an *equitable partition*. For a subset $V \subseteq \{1, 2, \dots, n\}$, the *characteristic vector* $\delta_V \in \mathbb{R}^n$ of V is the vector such that $\delta_V(i) = 1$ if $i \in V$ and 0 otherwise. The *characteristic matrix* P with respect to the partition Π is the $n \times m$ matrix whose jth column is the characteristic vector δ_{X_j} of X_j for $1 \leq j \leq m$, that is, $P = [\delta_{X_1} \mid \delta_{X_2} \mid \cdots \mid \delta_{X_m}]$. Therefore, it is not hard to verify that $MP = PB_m$. This fact implies the following result.

Lemma 2.1 ([8]): *Let M be a real symmetric matrix, and let Π be an equitable partition of M with quotient matrix B_m. Then we have*

(i) *all eigenvalues of B_m are also eigenvalues of M;*
(ii) *let x be an eigenvector of M corresponding to λ, if λ is not an eigenvalue of B_m then the summation of x on each cell is 0;*
(iii) *if y is an eigenvector of B_m corresponding to λ', then Py is an eigenvector of M corresponding to λ', where P is the characteristic matrix.*

Suppose that Γ is a connected graph of order n. Let $\mathcal{D}$ be the distance matrix of Γ and Π: $V = V_1 \cup \cdots \cup V_m$ a partition of the vertex set V. Suppose that $\mathcal{D}_{ij}$ is the submatrix of $\mathcal{D}$ whose rows and columns are induced by V_i and V_j for $1 \leq i, j \leq m$. For $u \in V_i$, the row sum of $\mathcal{D}_{ij}$ corresponding to u is $\sum_{v \in V_j} d(u, v)$. Thus, the partition Π is an equitable partition if, for $1 \leq i, j \leq m$ and $u \in V_i$, the value $\sum_{v \in V_j} d(u, v)$ is a constant independent of the choice of u. In this case, we say that Π is a *distance equitable partition* of Γ. Therefore, we get the following result by applying Lemma 2.1 to a distance matrix.

Corollary 2.2: *Let Γ be a connected graph with distance matrix $\mathcal{D}$ and Π a distance equitable partition of Γ with quotient matrix B_m of $\mathcal{D}$. Then we have*

(i) *all eigenvalues of B_m are distance eigenvalues of Γ;*
(ii) *let x be an eigenvector of $\mathcal{D}(\Gamma)$ corresponding to ∂, if ∂ is not an eigenvalue of B_m then the summation of x on each cell of Π is 0.*

Let $G \leq \mathrm{Aut}(\Gamma)$ be a subgroup of the automorphism group of the graph Γ. Suppose that $O_1, O_2, \ldots, O_k$ are all orbits of $V(\Gamma)$ under the action of G. Then Π_G: $V(\Gamma) = O_1 \cup O_2 \cup \cdots \cup O_k$ is a partition of $V(\Gamma)$ which is called the *orbit partition* of Γ on G. The following result confirms that an orbit partition is also a distance equitable partition.

Lemma 2.3 ([11, Lemma 2.1.]): *Let Γ be a connected graph and $G \leq \mathrm{Aut}(\Gamma)$. If Π_G : $V(\Gamma) = O_1 \cup O_2 \cup \cdots \cup O_k$ is the orbit partition of Γ with respect to G, then Π_G is also a distance equitable partition.*

3. Structure of $B(n, k)$

In this section, we first introduce some notations and symbols for $B(n, k)$ and give some basic structural properties of $B(n, k)$. Next, we present two types of distance equitable partitions of $B(n, k)$ according to its structure. For any subset $A \subseteq [n]$, let $\overline{A} = [n] \setminus A$.

Claim 1: *For two positive integers k and n such that $0 \leq k \leq n-1$, we have $B(n, k) \cong B(n, n-1-k)$.*

Proof: Let φ: $V(B(n, k)) \to V(B(n, n-1-k))$ be the map defined by $\varphi(S) = \overline{S}$ for any $S \in V(B(n, k))$. Clearly, φ is a bijection. Moreover, if $S_1 \sim S_2$ in $B(n, k)$ then $S_1 \subset S_2$ or $S_2 \subset S_1$. The former case leads to $\varphi(S_2) \subset \varphi(S_1)$, and the latter case leads to $\varphi(S_1) \subset \varphi(S_2)$, and thus $\varphi(S_1) \sim \varphi(S_2)$ in $B(n, n-1-k)$. Conversely, if $\varphi(S_1) \sim \varphi(S_2)$ in

$B(n, n-1-k)$, then $\varphi(S_1) \subset \varphi(S_2)$ or $\varphi(S_2) \subset \varphi(S_1)$. The former case leads to $S_2 \subset S_1$, and the latter case leads to $S_1 \subset S_2$, and thus $S_1 \sim S_2$ in $B(n,k)$. Therefore, φ is an isomorphism between $B(n,k)$ and $B(n, n-1-k)$. ■

If $k > (n-1)/2$ in the graph $B(n,k)$, then, by taking $k' = n-1-k$, the graph $B(n,k') \cong B(n,k)$ satisfies $k' \leq (n-1)/2$. Thus, we may always assume that $k \leq (n-1)/2$, i.e. $n \geq 2k+1$, for $B(n,k)$ throughout this paper.

Let $\mathcal{V}_k = \{S \subset [n] \mid |S| = k\}$ and $\mathcal{V}_{k+1} = \{S \subset [n] \mid |S| = k+1\}$ be the collections of k-subsets and the $k+1$-subsets of $[n]$, respectively. It is clear that $|\mathcal{V}_k| = \binom{n}{k}$, $|\mathcal{V}_{k+1}| = \binom{n}{k+1}$ and $V(B(n,k)) = \mathcal{V}_k \cup \mathcal{V}_{k+1}$ is a partition. Note that a bipartite graph Γ with partition $V_1 \cup V_2$ is called (r_1, r_2)-semi-regular if each vertex in V_1 has degree r_1 and each vertex in V_2 has degree r_2.

Claim 2: *The graph $B(n,k)$ is bipartite with two parts $\mathcal{V}_k$ and $\mathcal{V}_{k+1}$. Furthermore, $B(n,k)$ is $k+1$-regular if $n = 2k+1$ and is $(n-k, k+1)$-semi-regular if $n > 2k+1$.*

Proof: By the definition, $\mathcal{V}_k$ and $\mathcal{V}_{k+1}$ are two independent sets of $B(n,k)$, and thus $B(n,k)$ is bipartite. Moreover, each vertex $S = \{s_1, s_2, \ldots, s_k\} \in \mathcal{V}_k$ has $n-k$ neighbours $\{S \cup \{s\} \mid s \in \overline{S}\}$ in $\mathcal{V}_{k+1}$. Similarly, each vertex of $\mathcal{V}_{k+1}$ has $k+1$ neighbours in $\mathcal{V}_k$. It means that $B(n,k)$ is $k+1$-regular if $n = 2k+1$ and $(n-k, k+1)$-semi-regular if $n > 2k+1$. ■

For $A \in \mathcal{V}_k$ and $0 \leq i \leq k$, define

$$\mathcal{V}_{k,A}(i) = \{S \in \mathcal{V}_k \mid |S \cap A| = k - i\},$$
$$\mathcal{V}_{k+1,A}(i) = \{S \in \mathcal{V}_{k+1} \mid |S \cap A| = k - i\}.$$

Since $n \geq 2k+1$, we have $\mathcal{V}_{k,A}(i), \mathcal{V}_{k+1,A}(i) \neq \emptyset$ for $0 \leq i \leq k$. Particularly, $\mathcal{V}_{k,A}(0) = \{A\}$. By simple observations and calculations, we have the following claim.

Claim 3: *The cardinalities of $\mathcal{V}_{k,A}(i)$ and $\mathcal{V}_{k+1,A}(i)$ are*

$$|\mathcal{V}_{k,A}(i)| = \binom{k}{k-i}\binom{n-k}{i} \quad \text{and} \quad |\mathcal{V}_{k+1,A}(i)| = \binom{k}{k-i}\binom{n-k}{i+1}$$

for $0 \leq i \leq k$. Moreover, $\mathcal{V}_k = \bigcup_{i=0}^{k} \mathcal{V}_{k,A}(i)$ and $\mathcal{V}_{k+1} = \bigcup_{i=0}^{k} \mathcal{V}_{k+1,A}(i)$ for any $A \in V_k$.

In terms of these notations, we have the following result.

Lemma 3.1: *Let $A \in \mathcal{V}_k$ and $S \in V(B(n,k))$. Then, for any $0 \leq i \leq k$, the distance $d(S,A) = 2i$ if and only if $S \in \mathcal{V}_{k,A}(i)$, and $d(S,A) = 2i+1$ if and only if $S \in \mathcal{V}_{k+1,A}(i)$.*

Proof: Since $B(n,k)$ is bipartite with the partition $\mathcal{V}_k \cup \mathcal{V}_{k+1}$ and $A \in \mathcal{V}_k$, it is easy to see that $d(S,A)$ is even if and only if $S \in \mathcal{V}_k$ and $d(S,A)$ is odd if and only if $S \in \mathcal{V}_{k+1}$.

It is clear that $d(S,A) = 0$ if and only if $S = A \in \mathcal{V}_{k,A}(0) = \{A\}$. Moreover, $d(S,A) = 1$ if and only if $S = A \cup \{x\}$ for some $x \in \overline{A}$ if and only if $S \in \mathcal{V}_{k+1,A}(0) = \{A \cup \{x\} \mid x \in \overline{A}\}$. Therefore, the result follows for $i = 0$. In what follows, we consider the case of $i = l \geq 1$ and assume that the result holds for $i \leq l-1$.

First suppose that $d(S,A) = 2l$. We have $S \in \mathcal{V}_k$ and there exists $S' \in V(B(n,k))$ such that $S' \sim S$ and $d(S',A) = 2l - 1 = 2(l-1) + 1$. By inductive assumption, we have $S' \in \mathcal{V}_{k+1,A}(l-1)$, and thus, we may assume that $S' = \{x_1, \dots, x_{k-l+1}, x_{k-l+2}, \dots, x_{k+1}\}$, where $x_1, \dots, x_{k-l+1} \in A$ and $x_{k-l+2}, \dots, x_{k+1} \in \overline{A}$. Since $S' \sim S$, we have $S \subset S'$, that is, $S = S' \setminus \{x_j\}$ for some $1 \le j \le k+1$. If $k - l + 2 \le j$ then $S \in \mathcal{V}_{k,A}(l-1)$ and thus $d(S,A) = 2(l-1)$ by inductive assumption, a contradiction. Therefore, we have $j \le k - l + 1$ and thus $S \in \mathcal{V}_{k,A}(l)$. Conversely, suppose that $S \in \mathcal{V}_{k,A}(l)$. Since $l \ge 1$, there exists $x \in A \setminus S$. Therefore, the vertex $S' = S \cup \{x\}$ is adjacent to S. Note that $S' \in \mathcal{V}_{k+1,A}(l-1)$. We have $d(S,A) \le d(S,S') + d(S',A) = 1 + 2(l-1) + 1 = 2l$. If $d(S,A) = 2l' < 2l$ then we have $S \in \mathcal{V}_{k,A}(l')$ by inductive assumption, a contradiction. It follows that $d(S,A) = 2l$.

Next suppose that $d(S,A) = 2l + 1$. We have $S \in \mathcal{V}_{k+1}$ and thus there exists $S' \in \mathcal{V}_k$ such that $S' \sim S$ and $d(S',A) = 2l$. By the arguments of above segment, we have $S' \in \mathcal{V}_{k,A}(l)$ and $S' = S \setminus \{x_j\}$ for some $x_j \in S$. Thus, we may assume that $S' = \{x_1, \dots, x_{k-l}, x_{k-l+1}, \dots, x_k\}$, where $x_1, \dots, x_{k-l} \in A$ and $x_{k-l+1}, \dots, x_k \in \overline{A}$. We claim that $S \in \mathcal{V}_{k+1,A}(l)$, i.e. $|S \cap A| = k - l$. Otherwise, we have $x_j \in A$, which leads to $S \in \mathcal{V}_{k+1,A}(l-1)$ and thus $d(S,A) = 2(l-1) + 1 = 2l - 1$ by inductive assumption, a contradiction. Conversely, suppose that $S \in \mathcal{V}_{k+1,A}(l)$. Let $S' = S \setminus \{x\}$ for some $x \in S \setminus A$. We have $S' \sim S$ and $S' \in \mathcal{V}_{k,A}(l)$. Therefore, $d(S,A) \le d(S,S') + d(S',A) = 2l + 1$ by the arguments of the above segment. If $d(S,A) = 2l' + 1 < 2l + 1$, then $S \in \mathcal{V}_{k+1,A}(l')$ again by inductive assumption, a contradiction. It follows that $d(S,A) = 2l + 1$.

This completes the proof. ■

For $B \in \mathcal{V}_{k+1}$, let

$$\mathcal{V}^+_{k,B}(i) = \{S \in \mathcal{V}_k \mid |S \cap B| = k + 1 - i\} \quad \text{for } 1 \le i \le k+1,$$
$$\mathcal{V}^+_{k+1,B}(i) = \{S \in \mathcal{V}_{k+1} \mid |S \cap B| = k + 1 - i\} \quad \text{for } 0 \le i \le k+1.$$

Since $n \ge 2k + 1$, we have $\mathcal{V}^+_{k,B}(i), \mathcal{V}^+_{k+1,B}(j) \ne \emptyset$ for $1 \le i \le k+1$ and $0 \le j \le k$, and $\mathcal{V}^+_{k+1,B}(k+1) = \emptyset$ if $n = 2k+1$ and $\mathcal{V}^+_{k+1,B}(k+1) \ne \emptyset$ if $n > 2k+1$. As similar as Claim 3, we have the following claim.

Claim 4: *The cardinalities of $\mathcal{V}^+_{k,B}(i)$ and $\mathcal{V}^+_{k+1,B}(i)$ are*

$$|\mathcal{V}^+_{k,B}(i)| = \binom{k+1}{k+1-i}\binom{n-k-1}{i-1} \quad \text{and} \quad |\mathcal{V}^+_{k+1,B}(j)| = \binom{k+1}{k+1-j}\binom{n-k-1}{j}$$

for $1 \le i \le k+1$ and $0 \le j \le k+1$. Moreover, $\mathcal{V}_k = \bigcup_{i=1}^{k+1} \mathcal{V}^+_{k,B}(i)$ and $\mathcal{V}_{k+1} = \bigcup_{i=0}^{k+1} \mathcal{V}^+_{k+1,B}(i)$ for any $B \in \mathcal{V}_{k+1}$.

As similar to the proof of Lemma 3.1, we get the following result.

Lemma 3.2: *Let $B \in \mathcal{V}_{k+1}$ and $S \in V(B(n,k))$. Then the distance $d(S,B) = 2i$ if and only if $S \in \mathcal{V}^+_{k+1,B}(i)$ for any $0 \le i \le k+1$, and $d(S,B) = 2i - 1$ if and only if $S \in \mathcal{V}^+_{k,B}(i)$ for any $1 \le i \le k+1$.*

For any $A \in \mathcal{V}_k$ and $B \in \mathcal{V}_{k+1}$, Lemma 3.1 means that the longest path from A to other vertices of $B(n,k)$ has length $2k+1$, and Lemma 3.2 means that the longest path from B to other vertices of $B(n,k)$ has length $2k+1$ if $n = 2k+1$ and has length $2k+2$ if $n > 2k+1$. Therefore, we have the following result.

Corollary 3.3: *Let n and k be two positive integers such that $n \geq 2k+1$. The graph $B(n,k)$ has diameter $2k+1$ if $n = 2k+1$ and has diameter $2k+2$ if $n > 2k+1$.*

Let S_n be the symmetric group on the set $[n]$. For each $\sigma \in S_n$, let $\tilde{\sigma}\colon V(B(n,k)) \to V(B(n,k))$ be the map defined by $\tilde{\sigma}(S) = \{\sigma(s) \mid s \in S\}$ for $S \in V(B(n,k))$ and let $\tilde{S}_n = \{\tilde{\sigma} \mid \sigma \in S_n\}$. We get the following result.

Claim 5: *The map $\tilde{\sigma} \in \mathrm{Aut}(B(n,k))$ and thus $\tilde{S}_n \leq \mathrm{Aut}(B(n,k))$. Moreover, $\mathcal{V}_k$ and $\mathcal{V}_{k+1}$ are orbits on $\tilde{S}_n$.*

Proof: By the definition, one can easily verify that $\tilde{\sigma} \in \mathrm{Aut}(B(n,k))$ and $\tilde{S}_n \leq \mathrm{Aut}(B(n,k))$. For any two vertices $S = \{s_1, \ldots, s_k\}$ and $S' = \{s'_1, \ldots, s'_k\}$ of $\mathcal{V}_k$, by taking $\sigma \in S_n$ such that $\sigma(s_i) = s'_i$ for $1 \leq i \leq k$, we have $\tilde{\sigma}(S) = S'$. Therefore, $\mathcal{V}_k$ is an orbit. Similarly, $\mathcal{V}_{k+1}$ is also an orbit. ■

Claim 6: *The partition $\Pi\colon V(B(n,k)) = \mathcal{V}_k \cup \mathcal{V}_{k+1}$ is a distance equitable partition of $B(n,k)$ with quotient matrix*

$$\mathcal{D}_\Pi = \begin{pmatrix} \alpha_{11}(n,k) & \alpha_{12}(n,k) \\ \alpha_{21}(n,k) & \alpha_{22}(n,k) \end{pmatrix}$$

where $\alpha_{11}(n,k) = 2k\binom{n-1}{k}$, $\alpha_{12}(n,k) = 2k\binom{n-1}{k+1} + \binom{n}{k+1}$, $\alpha_{21}(n,k) = 2(k+1)\binom{n-1}{k} - \binom{n}{k}$ and $\alpha_{22}(n,k) = 2(k+1)\binom{n-1}{k+1}$.

Proof: From Claim 5, Π is an orbit partition of $B(n,k)$. Therefore, Lemma 2.3 indicates that Π is a distance equitable partition of $B(n,k)$. Assume that the quotient matrix is given by

$$\mathcal{D}_\Pi = \begin{pmatrix} \alpha_{11}(n,k) & \alpha_{12}(n,k) \\ \alpha_{21}(n,k) & \alpha_{22}(n,k) \end{pmatrix}.$$

For any $A \in \mathcal{V}_k$ and $B \in \mathcal{V}_{k+1}$, by Claims 3, 4 and Lemmas 3.1 and 3.2, we have

$$\begin{aligned}
\alpha_{11}(n,k) &= \sum_{S \in \mathcal{V}_k} d(S,A) = \sum_{i=0}^{k} \sum_{S \in \mathcal{V}_{k,A}(i)} d(S,A) \\
&= \sum_{i=0}^{k} \sum_{S \in \mathcal{V}_{k,A}(i)} 2i = \sum_{i=0}^{k} 2i \binom{k}{k-i}\binom{n-k}{i} = 2k\binom{n-1}{k},
\end{aligned}$$

$$\begin{aligned}
\alpha_{12}(n,k) &= \sum_{S \in \mathcal{V}_{k+1}} d(S,A) = \sum_{i=0}^{k} \sum_{S \in \mathcal{V}_{k+1,A}(i)} d(S,A) \\
&= \sum_{i=0}^{k} \sum_{S \in \mathcal{V}_{k,A}(i)} (2i+1) = \sum_{i=0}^{k} (2i+1) \binom{k}{k-i}\binom{n-k}{i+1}
\end{aligned}$$

$$= 2k\binom{n-1}{k+1} + \binom{n}{k+1},$$

$$\alpha_{21}(n,k) = \sum_{S\in\mathcal{V}_k} d(S,B) = \sum_{i=1}^{k+1} \sum_{S\in\mathcal{V}^+_{k,B}(i)} d(S,B),$$

$$= \sum_{i=1}^{k+1} \sum_{S\in\mathcal{V}^+_{k,B}(i)} (2i-1) = \sum_{i=1}^{k+1}(2i-1)\binom{k+1}{k+1-i}\binom{n-k}{i-1}$$

$$= 2(k+1)\binom{n-1}{k} - \binom{n}{k},$$

$$\alpha_{22}(n,k) = \sum_{S\in\mathcal{V}_{k+1}} d(S,B) = \sum_{i=0}^{k+1} \sum_{S\in\mathcal{V}^+_{k+1,B}(i)} d(S,B)$$

$$= \sum_{i=0}^{k+1} \sum_{S\in\mathcal{V}^+_{k+1,B}(i)} 2i = \sum_{i=0}^{k+1} 2i\binom{k+1}{k+1-i}\binom{n-k-1}{i} = 2(k+1)\binom{n-1}{k+1}. \blacksquare$$

By Claim 6, the eigenvalues of $\mathcal{D}_\Pi$, denoted by $\theta_1(n,k)$ and $\theta_2(n,k)$, are the roots of the quadratic equation

$$f(x) = \det(xI - \mathcal{D}_\Pi) = x^2 - 2(n-1)\binom{n-1}{k}x - \binom{n}{k}\binom{n}{k+1} = 0. \qquad (1)$$

By simple calculations, we have

$$\theta_1(n,k), \theta_2(n,k) = \binom{n-1}{k}\left((n-1) \pm \sqrt{(n-1)^2 + \frac{n^2}{(n-k)(k+1)}}\right). \qquad (2)$$

From Corollary 2.2, we get the following result.

Lemma 3.4: *Let $\theta_1(n,k)$ and $\theta_2(n,k)$ be the two roots of Equation* (1) *given in Equation* (2). *Then we have*

(i) *$\theta_1(n,k)$ and $\theta_2(n,k)$ are distance eigenvalues of $B(n,k)$;*
(ii) *if x is an eigenvector of $\mathcal{D}(B(n,k))$ corresponding to λ different from θ_1 and θ_2, then $\sum_{v\in\mathcal{V}_k} x(v) = \sum_{v\in\mathcal{V}_{k+1}} x(v) = 0$.*

For given $a \in [n]$, let $G_a = \{\sigma \in S_n \mid \sigma(a) = a\}$ be the stabilizer of a. Clearly, $G_a \leq S_n$ and thus $\tilde{G}_a = \{\tilde{\sigma} \mid \sigma \in G_a\} \leq \tilde{S}_n \leq Aut(B(n,k))$. As similar as Claim 5, it is easy to see that $\mathcal{V}_{a,k}$, $\mathcal{V}_{a,k+1}$, $\mathcal{V}_{\bar{a},k}$ and $\mathcal{V}_{\bar{a},k+1}$ are orbits on $\tilde{G}_a$, where

$$\mathcal{V}_{a,k} = \{S \in \mathcal{V}_k \mid a \in S\}, \quad \mathcal{V}_{a,k+1} = \{S \in \mathcal{V}_{k+1} \mid a \in S\},$$
$$\mathcal{V}_{\bar{a},k} = \{S \in \mathcal{V}_k \mid a \notin S\}, \quad \mathcal{V}_{\bar{a},k+1} = \{S \in \mathcal{V}_{k+1} \mid a \notin S\}.$$

As similar as Claim 6, we have the following result.

Claim 7: *The partition* $\Pi_a : V(B(n,k)) = \mathcal{V}_{a,k} \cup \mathcal{V}_{a,k+1} \cup \mathcal{V}_{\overline{a},k} \cup \mathcal{V}_{\overline{a},k+1}$ *is also a distance equitable partition of* $B(n,k)$ *with quotient matrix*

$$\mathcal{D}_{\Pi_a} = \begin{pmatrix} \alpha_{11}(n-1,k-1) & \alpha_{12}(n-1,k-1) & \alpha_{11}(n-1,k)+2\binom{n-2}{k-1} & \alpha_{12}(n-1,k)+2\binom{n-2}{k} \\ \alpha_{21}(n-1,k-1) & \alpha_{22}(n-1,k-1) & \alpha_{21}(n-1,k)+2\binom{n-2}{k-1} & \alpha_{22}(n-1,k)+2\binom{n-2}{k} \\ \alpha_{11}(n-1,k-1)+2\binom{n-2}{k-1} & \alpha_{12}(n-1,k-1)+2\binom{n-2}{k} & \alpha_{11}(n-1,k) & \alpha_{12}(n-1,k) \\ \alpha_{21}(n-1,k-1)+2\binom{n-2}{k-1} & \alpha_{22}(n-1,k-1)+2\binom{n-2}{k} & \alpha_{21}(n-1,k) & \alpha_{22}(n-1,k) \end{pmatrix}.$$

where $\alpha_{ij}(\cdot,\cdot)$ *is given in Claim 6.*

Proof: Since Π_a is an orbit partition of $B(n,k)$, Lemma 2.3 means that Π_a is also a distance equitable partition of $B(n,k)$. Denote the quotient matrix by

$$\mathcal{D}_{\Pi_a} = \begin{pmatrix} d(\mathcal{V}_{a,k},\mathcal{V}_{a,k}) & d(\mathcal{V}_{a,k},\mathcal{V}_{a,k+1}) & d(\mathcal{V}_{a,k},\mathcal{V}_{\overline{a},k}) & d(\mathcal{V}_{a,k},\mathcal{V}_{\overline{a},k+1}) \\ d(\mathcal{V}_{a,k+1},\mathcal{V}_{a,k}) & d(\mathcal{V}_{a,k+1},\mathcal{V}_{a,k+1}) & d(\mathcal{V}_{a,k+1},\mathcal{V}_{\overline{a},k}) & d(\mathcal{V}_{a,k+1},\mathcal{V}_{\overline{a},k+1}) \\ d(\mathcal{V}_{\overline{a},k},\mathcal{V}_{a,k}) & d(\mathcal{V}_{\overline{a},k},\mathcal{V}_{a,k+1}) & d(\mathcal{V}_{\overline{a},k},\mathcal{V}_{\overline{a},k}) & d(\mathcal{V}_{\overline{a},k},\mathcal{V}_{\overline{a},k+1}) \\ d(\mathcal{V}_{\overline{a},k+1},\mathcal{V}_{a,k}) & d(\mathcal{V}_{\overline{a},k+1},\mathcal{V}_{a,k+1}) & d(\mathcal{V}_{\overline{a},k+1},\mathcal{V}_{\overline{a},k}) & d(\mathcal{V}_{\overline{a},k+1},\mathcal{V}_{\overline{a},k+1}) \end{pmatrix}.$$

For each $A \in V_{a,k}$, by Lemma 3.1, we have

$$d(\mathcal{V}_{a,k},\mathcal{V}_{a,k}) = \sum_{S\in\mathcal{V}_{a,k}} d(S,A) = \sum_{i=0}^{k} \sum_{S\in\mathcal{V}_{k,A}(i),a\in S} d(S,A) = \sum_{i=0}^{k} \sum_{S\in\mathcal{V}_{k,A}(i),a\in S} 2i$$
$$= \sum_{i=0}^{k} 2i\binom{k-1}{k-1-i}\binom{n-k}{i} = 2(k-1)\binom{n-2}{k-1} = \alpha_{11}(n-1,k-1),$$

$$d(\mathcal{V}_{a,k},\mathcal{V}_{a,k+1}) = \sum_{S\in\mathcal{V}_{a,k+1}} d(S,A) = \sum_{i=0}^{k} \sum_{S\in\mathcal{V}_{k+1,A}(i),a\in S} d(S,A) = \sum_{i=0}^{k} \sum_{S\in\mathcal{V}_{k+1,A}(i),a\in S} (2i+1)$$
$$= \sum_{i=0}^{k} (2i+1)\binom{k-1}{k-1-i}\binom{n-k}{i+1}$$
$$= 2(k-1)\binom{n-2}{k} + \binom{n-1}{k} = \alpha_{12}(n-1,k-1),$$

$$d(\mathcal{V}_{a,k},\mathcal{V}_{\overline{a},k}) = \sum_{S\in\mathcal{V}_{\overline{a},k}} d(S,A) = \sum_{i=0}^{k} \sum_{S\in\mathcal{V}_{k,A}(i),a\notin S} d(S,A) = \sum_{i=0}^{k} \sum_{S\in V_{k,A}(i),a\notin\in S} 2i$$
$$= \sum_{i=0}^{k} 2i\binom{k-1}{k-i}\binom{n-k}{i} = 2k\binom{n-2}{k}$$
$$+2\binom{n-2}{k-1} = \alpha_{11}(n-1,k) + 2\binom{n-2}{k-1},$$

$$d(\mathcal{V}_{a,k},\mathcal{V}_{\bar{a},k+1}) = \sum_{S\in\mathcal{V}_{\bar{a},k+1}} d(S,A) = \sum_{i=0}^{k}\sum_{S\in\mathcal{V}_{k+1,A}(i),a\notin S} d(S,A) = \sum_{i=0}^{k}\sum_{S\in\mathcal{V}_{k+1,A}(i),a\notin S}(2i+1)$$
$$= \sum_{i=0}^{k}(2i+1)\binom{k-1}{k-i}\binom{n-k}{i+1}$$
$$= 2(k-1)\binom{n-2}{k+1} + 3\binom{n-1}{k+1} = \alpha_{12}(n-1,k) + 2\binom{n-2}{k+1}.$$

Similarly, other entries of $\mathcal{D}_{\Pi_a}$ are obtained by Lemmas 3.1 and 3.2. ■

To end up this part, we obtain all distinct eigenvalues of $\mathcal{D}_{\Pi_a}$.

Lemma 3.5: *Let n and k be two positive integers such that $n \geq 2k+1$. Then we have*

(i) *the eigenvalues of $\mathcal{D}_{\Pi_a}$ are $\theta_1(n,k)$, $\theta_2(n,k)$, 0 and $-2\binom{n-1}{k}$, which are distance eigenvalues of $B(n,k)$;*
(ii) *if x is an eigenvector of $\mathcal{D}(B(n,k))$ corresponding to λ different from θ_1, θ_2, 0 or $-2\binom{n-1}{k}$, then $\sum_{v\in\mathcal{V}_{a,k}} x(v) = \sum_{v\in\mathcal{V}_{a,k+1}} x(v) = \sum_{v\in\mathcal{V}_{\bar{a},k}} x(v) = \sum_{v\in\mathcal{V}_{\bar{a},k+1}} x(v) = 0$ for any $a \in [n]$.*

Proof: Let $y = (y_1, y_2)^T$ be an eigenvector of $\mathcal{D}(\Pi)$ corresponding to the eigenvalue $\theta_i(n,k)$ for some $i \in \{1,2\}$. Let P be the characteristic matrix of the partition Π. Therefore, Lemma 2.1(iii) indicates that Py is an eigenvector of $\mathcal{D}(B(n,k))$ corresponding to $\theta_i(n,k)$. Note that $y^* = Py \in \mathbb{R}^{|V(B(n,k))|}$ is such that $y^*(v) = y_1$ if $v \in \mathcal{V}_k$ and $y^*(v) = y_2$ if $v \in \mathcal{V}_{k+1}$. Therefore, $\sum_{v\in\mathcal{V}_{a,k}} y^*(v) = \binom{n-1}{k-1}y_1 \neq 0$ for any $a \in [n]$, and Corollary 2.2(ii) implies that $\theta_i(n,k)$ is an eigenvalue of $\mathcal{D}_{\Pi_a}$. From Claim 7, one can easily verify that $\mathrm{row}_1(\mathcal{D}_{\Pi_a}) + \mathrm{row}_3(\mathcal{D}_{\Pi_a}) = \mathrm{row}_2(\mathcal{D}_{\Pi_a}) + \mathrm{row}_4(\mathcal{D}_{\Pi_a})$. It follows that 0 is an eigenvalue of $\mathcal{D}_{\Pi_a}$. Assume that the remaining eigenvalue of $\mathcal{D}(\Pi_a)$ is ϵ. It holds that

$$\theta_1(n,k) + \theta_2(n,k) + 0 + \epsilon = \mathrm{tr}(\mathcal{D}_{\Pi_a}) = \alpha_{11}(n-1,k-1) + \alpha_{22}(n-1,k-1)$$
$$+ \alpha_{11}(n-1,k) + \alpha_{22}(n-1,k).$$

By calculations, we have $\epsilon = -2\binom{n-1}{k}$ and thus all eigenvalues of $\mathcal{D}_{\Pi_a}$ are $\theta_1(n,k), \theta_2(n,k)$, 0, and $-2\binom{n-1}{k}$. Thus, (i) follows.

Assume that x is an eigenvector of $\mathcal{D}(B(n,k))$ corresponding to λ different from θ_1, θ_2, 0 or $-2\binom{n-1}{k}$. For any $a \in [n]$, since λ is not an eigenvalue of $\mathcal{D}_{\Pi_a}$, Corollary 2.2(iii) means that x sums to zero on each cell of the partition Π_a. Thus, (ii) follows. ■

4. Distance spectrum of $B(n,k)$

In this part, we first obtain all distinct distance eigenvalues of $B(n,k)$ and next determine the multiplicities of them. Let A be a set and m: $A \to \mathbb{N}^+$ a function. The *multiset* A^m with *underlying set* A and *multiplicity function* m is the collection of all elements of A and each element $a \in A$ occurs $m(a)$ times, that is, $A^m = \{m(a) * a \mid a \in A\}$ where $m(a)$ counts the number of occurrences of a. Additionally, we may extend the domain of m by defining $m(x) = 0$ for each $x \notin A$. Clearly, a set A can be viewed as a multiset A^m with $m(a) = 1$ for any $a \in A$. For two multisets A^{m_1} and B^{m_2}, the *multi-union* of them, denote by $A^{m_1} \uplus B^{m_2}$, is the multiset with the underlying set $A \cup B$ and multiplicity function m, where $m(\eta) = m_1(\eta) + m_2(\eta)$ for any $\eta \in A \cup B$. Particularly, let $k * A^m$ denote multi-union of k copies of A^m. It is clear that $j * \mathcal{V}_{k,A}(i) = \{j * S \mid S \in \mathcal{V}_k, |S \cap A| = k - i\}$ is the multiset with multiplicity function m such that $m(S) = j$ for any $S \in \mathcal{V}_{k,A}(i)$. Similarly, the multiset $j * \mathcal{V}_{k+1,A}(i)$ is well-defined. Of course, j may equal i. We get the following result.

Lemma 4.1: *Let n and k be two positive integers such that $n \geq 2k + 1$.*

(1) *If $A \in \mathcal{V}_k$, then $\biguplus_{i=1}^{k} i * \mathcal{V}_{k,A}(i) = \biguplus_{a\in A} \mathcal{V}_{\overline{a},k}$ and $\biguplus_{i=1}^{k} i * \mathcal{V}_{k+1,A}(i) = \biguplus_{a\in A} \mathcal{V}_{\overline{a},k+1}$.*
(2) *If $B \in \mathcal{V}_{k+1}$ then $\biguplus_{i=1}^{k+1} i * \mathcal{V}^{+}_{k,B}(i) = \biguplus_{b\in B} \mathcal{V}_{\overline{b},k}$ and $\biguplus_{i=1}^{k+1} i * \mathcal{V}^{+}_{k+1,B}(i) = \biguplus_{b\in B} \mathcal{V}_{\overline{b},k+1}$.*

Proof: Let $\mathcal{A}^{m_1}$ and $\mathcal{B}^{m_2}$ denote the multiset $\biguplus_{i=1}^{k} i * \mathcal{V}_{k,A}(i)$ and $\biguplus_{a\in A} \mathcal{V}_{\overline{a},k}$, respectively. Notice that each $S \in \mathcal{V}_{k,A}(i)$ is distinct, it is clear that $\mathcal{A} = \bigcup_{i=1}^{k} \mathcal{V}_{k,A}(i) = \mathcal{V}_k \setminus \mathcal{V}_{k,A}(0) = \mathcal{V}_k \setminus \{A\}$. Also by noticing that any k-element subset of $\mathcal{V}_k$ other than A is contained in $\biguplus_{a\in A} \mathcal{V}_{\overline{a},k}$, we have $\mathcal{B} = \mathcal{V}_k \setminus \{A\}$. If $S \in \mathcal{V}_k \setminus \{A\}$ then $S \in \mathcal{V}_{k,A}(i)$ for some $1 \leq i \leq k$. Clearly, $S \in \mathcal{A}$ only occurs in $\mathcal{V}_{k,A}(i)$ and thus the multiplicity of S in $\mathcal{A}^{m_1}$ equals $m_1(S) = i$. Now we may assume that $S = \{s_1, \ldots, s_{k-i}, s_{k-i+1}, \ldots, s_k\}$ where $s_1, \ldots, s_{k-i} \in A, s_{k-i+1}, \ldots, s_k \in \overline{A}$ and $i \geq 1$. It is easy to see that, for $a \in A \setminus \{s_1, \ldots, s_{k-i}\}$, $S \in \mathcal{B}$ occurs exactly one time in each $\mathcal{V}_{\overline{a},k}$ and thus $m_2(S) = i$ due to $|A \setminus \{s_1, \ldots, s_{k-i}\}| = i$. Therefore, we have $\mathcal{A}^{m_1} = \mathcal{B}^{m_2}$, that is $\biguplus_{i=1}^{k} i * \mathcal{V}_{k,A}(i) = \biguplus_{a\in A} \mathcal{V}_{\overline{a},k}$. Similarly, one can also obtain that $\biguplus_{i=1}^{k} i * \mathcal{V}_{k+1,A}(i) = \biguplus_{a\in A} \mathcal{V}_{\overline{a},k+1}$ and (1) holds.

Let $\mathcal{F}^{m_3} = \biguplus_{i=1}^{k+1} i * \mathcal{V}^{+}_{k,B}(i)$ and $\mathcal{L}^{m_4} = \biguplus_{b\in B} \mathcal{V}_{\overline{b},k}$. It is easy to see that $\mathcal{F} = \mathcal{V}_k$. For each $S \in \mathcal{V}_k$, there exists $b \in B$ such that $b \notin S$ and thus $S \in \mathcal{V}_{\overline{b},k}$. Therefore, we also have $\mathcal{L} = \mathcal{V}_k$. Let S be an element of $\mathcal{V}_k$. Assume that $S = \{s_1, \ldots, s_{k+1-i}, s_{k+2-i}, \ldots, s_k\} \in \mathcal{V}^{+}_{k,B}(i)$ where $s_1, \ldots, s_{k+1-i} \in B$ and $s_{k+2-i}, \ldots, s_k \in \overline{B}$. Since S does not occur in $\mathcal{V}^{+}_{k,B}(j)$ for any $j \neq i$, it holds that $m_3(S) = i$. Note that, for $b \in B$, $S \in \mathcal{V}_{\overline{b},k}$ if and only if $b \in B \setminus \{s_1, \ldots, s_{k+1-i}\}$. Therefore, $m_4(S) = |B \setminus \{s_1, \ldots, s_{k+1-i}\}| = i$. Thus, $\mathcal{F}^{m_3} = \mathcal{L}^{m_4}$, that is, $\biguplus_{i=1}^{k+1} i * \mathcal{V}^{+}_{k,B}(i) = \biguplus_{b\in B} \mathcal{V}_{\overline{b},k}$. Similarly, one can also obtain that $\biguplus_{i=1}^{k+1} i * \mathcal{V}^{+}_{k+1,B}(i) = \biguplus_{b\in B} \mathcal{V}_{\overline{b},k+1}$ and (2) holds. ■

Let Γ be a graph and x: $V(\Gamma) \to \mathbb{R}$ a real-valued function on $V(\Gamma)$. For a multiset A^m with $A \subseteq V(\Gamma)$, denote by $x(A^m) = \sum_{a\in A^m} x(a) = \sum_{a\in A} m(a)x(a)$. Now we are ready to give one of our main results.

Theorem 4.2: *Let n and k be two positive integers such that $n \geq 2k + 1$. Then the distinct distance eigenvalues of $B(n, k)$ are $\theta_1(n, k)$, $\theta_2(n, k)$, 0 and $-2\binom{n-1}{k}$.*

Proof: Suppose to the contrary that $B(n, k)$ has a distance eigenvalue ∂ different from $\theta_1(n, k)$, $\theta_2(n, k)$, 0 and $-2\binom{n-1}{k}$. Let x be an eigenvector of $\mathcal{D}(B(n, k))$ corresponding to ∂. By Lemma 3.5 (ii), we have $x(\mathcal{V}_{a,k}) = x(\mathcal{V}_{a,k+1}) = x(\mathcal{V}_{\overline{a},k}) = x(\mathcal{V}_{\overline{a},k+1}) = 0$ for any $a \in [n]$. It follows that $x(\mathcal{V}_k) = x(\mathcal{V}_{a,k}) + x(\mathcal{V}_{\overline{a},k}) = 0$ and $x(\mathcal{V}_{k+1}) = x(\mathcal{V}_{a,k+1}) + x(\mathcal{V}_{\overline{a},k+1}) = 0$.

Note that $x \neq 0$. There exists $A \in \mathcal{V}_k$ such that $x(A) \neq 0$ or $B \in \mathcal{V}_{k+1}$ such that $x(B) \neq 0$. We only consider the former case since the latter one is similar. Since $\mathcal{D}(B(n,k))x = \partial x$, we have $\partial x(A) = \sum_{S \in V(B(n,k))} d(S,A)x(S)$. Now we partition the vertex set $V(B(n,k)$ according to the distance from A. By Lemmas 3.1 and 3.2, we see that $\{S \in V(B(n,k)) \mid d(S,A) = 1\} = \mathcal{V}_{k+1,A}(0)$, $\{S \in V(B(n,k)) \mid d(S,A) = 2\} = \mathcal{V}_{k,A}(1)$, $\{S \in V(B(n,k)) \mid d(S,A) = 3\} = \mathcal{V}_{k+1,A}(1)$ and, in general, $\{S \in V(B(n,k)) \mid d(S,A) = 2i\} = \mathcal{V}_{k,A}(i)$ and $\{S \in V(B(n,k)) \mid d(S,A) = 2i+1\} = \mathcal{V}_{k+1,A}(i)$ for $0 \leq i \leq k$. Thus, we have

$$\begin{aligned}
\partial x(A) &= \sum_{S \in V(B(n,k))} d(S,A)x(S) = \sum_{S \in \mathcal{V}_k} d(S,A)x(S) + \sum_{S \in \mathcal{V}_{k+1}} d(S,A)x(S) \\
&= x(\mathcal{V}_{k+1,A}(0)) + 2 \cdot x(\mathcal{V}_{k,A}(1)) + 3 \cdot x(\mathcal{V}_{k+1,A}(1)) \\
&\quad + \cdots + 2i \cdot x(\mathcal{V}_{k,A}(i)) + (2i+1) \cdot x(\mathcal{V}_{k+1,A}(i)) + \cdots \\
&\quad + 2k \cdot x(\mathcal{V}_{k,A}(k)) + (2k+1) \cdot x(\mathcal{V}_{k+1,A}(k)) \\
&= x(\mathcal{V}_{k+1,A}(0)) + x(\mathcal{V}_{k+1,A}(1)) + x(\mathcal{V}_{k+1,A}(2)) + \cdots + x(\mathcal{V}_{k+1,A}(k)) \\
&\quad + 2(x(\mathcal{V}_{k+1,A}(1)) + 2x(\mathcal{V}_{k+1,A}(2)) + \cdots \\
&\quad + ix(\mathcal{V}_{k+1,A}(i)) + \cdots + kx(\mathcal{V}_{k+1,A}(k))) \\
&\quad + 2(x(\mathcal{V}_{k,A}(1)) + 2x(\mathcal{V}_{k,A}(2)) + \cdots + ix(\mathcal{V}_{k,A}(i)) + \cdots + kx(\mathcal{V}_{k,A}(k))) \\
&= x\left(\bigcup_{i=0}^{k} \mathcal{V}_{k+1,A}(i)\right) + 2x\left(\biguplus_{i=1}^{k} i * \mathcal{V}_{k+1,A}(i)\right) + 2x\left(\biguplus_{i=1}^{k} i * \mathcal{V}_{k,A}(i)\right).
\end{aligned}$$

Note that $\bigcup_{i=0}^{k} \mathcal{V}_{k+1,A}(i) = \mathcal{V}_{k+1}$, and Lemma 4.1(1) gives $\biguplus_{i=1}^{k} i * \mathcal{V}_{k+1,A}(i) = \biguplus_{a \in A} \mathcal{V}_{\bar{a},k+1}$ and $\biguplus_{i=1}^{k} i * \mathcal{V}_{k,A}(i) = \biguplus_{a \in A} \mathcal{V}_{\bar{a},k}$. Therefore, we have

$$\begin{aligned}
\partial x(A) = x(\mathcal{V}_{k+1}) + 2x\left(\biguplus_{a \in A} \mathcal{V}_{\bar{a},k+1}\right) + 2x\left(\biguplus_{a \in A} \mathcal{V}_{\bar{a},k}\right) = x(\mathcal{V}_{k+1}) \\
+ 2\sum_{a \in A} x(\mathcal{V}_{\bar{a},k+1}) + 2\sum_{a \in A} x(\mathcal{V}_{\bar{a},k}) = 0.
\end{aligned}$$

It leads to that $\partial = 0$, a contradiction. ■

All distinct distance eigenvalues of $B(n,k)$ are obtained in Theorem 4.2. To determine the multiplicities of these eigenvalues, we need the following result.

Lemma 4.3: *The trace of $\mathcal{D}(B(n,k))^2$ is given by*

$$\operatorname{tr}(\mathcal{D}(B(n,k))^2) = 4n(n-1)\binom{n-1}{k}^2 + 2\binom{n}{k}\binom{n}{k+1}.$$

Proof: For $A \in \mathcal{V}_k$, by Claim 3 and Lemma 3.1, the diagonal entry of $\mathcal{D}(B(n,k))^2$ corresponding to A is given by

$$\begin{aligned}
d^{(2)}(A,A) &= \sum_{S\in V(B(n,k))} d(S,A)^2 = \sum_{S\in\mathcal{V}_k} d(S,A)^2 + \sum_{S\in\mathcal{V}_{k+1}} d(S,A)^2 \\
&= \sum_{i=0}^{k} \sum_{S\in\mathcal{V}_{k,A}(i)} d(S,A)^2 + \sum_{i=0}^{k} \sum_{S\in\mathcal{V}_{k+1,A}(i)} d(S,A)^2 \\
&= \sum_{i=0}^{k} \binom{k}{k-i}\binom{n-k}{i}(2i)^2 + \sum_{i=0}^{k} \binom{k}{k-i}\binom{n-k}{i+1}(2i+1)^2 \\
&= 4\sum_{i=0}^{k} i\binom{k}{i}\cdot i\binom{n-k}{i} \\
&\quad + \left(4\sum_{i=0}^{k} i\binom{k}{i}\cdot(i+1)\binom{n-k}{i+1} + \sum_{i=0}^{k}\binom{k}{k-i}\binom{n-k}{i+1}\right) \\
&= 4\sum_{i=0}^{k} k\binom{k-1}{i-1}\cdot(n-k)\binom{n-k-1}{i-1} \\
&\quad + \left(4\sum_{i=0}^{k} k\binom{k-1}{i-1}\cdot(n-k)\binom{n-k-1}{i} + \binom{n}{k+1}\right) \\
&= 4k(n-k)\sum_{i=0}^{k}\binom{k-1}{k-i}\binom{n-k-1}{i-1} \\
&\quad + 4k(n-k)\sum_{i=0}^{k}\binom{k-1}{k-i}\binom{n-k-1}{i} + \binom{n}{k+1} \\
&= 4k(n-k)\binom{n-2}{k-1} + 4k(n-k)\binom{n-2}{k} + \binom{n}{k+1} \\
&= 4k(n-k)\binom{n-1}{k} + \binom{n}{k+1}.
\end{aligned}$$

Similarly, for $B \in \mathcal{V}_{k+1}$, by Claim 4 and Lemma 3.2, the diagonal entry of $\mathcal{D}(B(n,k))^2$ corresponding to B is given by

$$d^{(2)}(B,B) = 4(k+1)(n-k-1)\binom{n-1}{k} + \binom{n}{k}.$$

Therefore, the trace of $\mathcal{D}(B(n,k))^2$ is given by

$$\begin{aligned}
\operatorname{tr}(\mathcal{D}(B(n,k))^2) &= |\mathcal{V}_k| d^{(2)}(A,A) + |\mathcal{V}_{k+1}| d^{(2)}(B,B) \\
&= 4k(n-k)\binom{n}{k}\binom{n-1}{k} + \binom{n}{k}\binom{n}{k+1}
\end{aligned}$$

$$+ 4(k+1)(n-k-1)\binom{n-1}{k}\binom{n}{k+1} + \binom{n}{k}\binom{n}{k+1}$$
$$= 4kn\binom{n-1}{k}^2 + 4n(n-k-1)\binom{n-1}{k}^2 + 2\binom{n}{k}\binom{n}{k+1}$$
$$= 4n(n-1)\binom{n-1}{k}^2 + 2\binom{n}{k}\binom{n}{k+1}.$$

It completes the proof. ■

Now we are ready to give the distance spectrum of $B(n,k)$.

Theorem 4.4: *Let n and k be two positive integers such that $n \geq 2k+1$. Then the distance spectrum of $B(n,k)$ is given by*

$$\mathrm{Sp}_{\mathcal{D}}(B(n,k)) = \left\{\theta_1(n,k), [0]^{\zeta}, \theta_2(n,k), \left[-2\binom{n-1}{k}\right]^{n-1}\right\}$$

where $\theta_1(n,k), \theta_2(n,k) = \binom{n-1}{k}((n-1) \pm \sqrt{(n-1)^2 + n^2/(n-k)(k+1)})$ *and* $\zeta = \binom{n+1}{k+1} - (n+1)$.

Proof: It is proved that $\theta_1(n,k), \theta_2(n,k), 0$ and $-2\binom{n-1}{k}$ are all distinct distance eigenvalues of $B(n,k)$ in Theorem 4.2. In what follows, we respectively write θ_1 and θ_2 for $\theta_1(n,k)$ and $\theta_2(n,k)$ for short. Note that the distance spectral radius θ_1 is simple. Assume that the multiplicities of θ_2 and $-2\binom{n-1}{k}$ are m_1 and m_2, respectively. Therefore, Lemma 4.3 implies that

$$\theta_1 + m_1\theta_2 - 2m_2\binom{n-1}{k} = \mathrm{tr}(\mathcal{D}(B(n,k))) = 0,$$
$$\theta_1^2 + m_1\theta_2^2 + 4m_2\binom{n-1}{k}^2 = \mathrm{tr}(\mathcal{D}(B(n,k))^2)$$
$$= 4n(n-1)\binom{n-1}{k}^2 + 2\binom{n}{k}\binom{n}{k+1}. \tag{3}$$

It leads to

$$\left[2\binom{n-1}{k}\theta_2 + \theta_2^2\right] m_1 = 4n(n-1)\binom{n-1}{k}^2 + 2\binom{n}{k}\binom{n}{k+1} - \theta_1^2 - 2\binom{n-1}{k}\theta_1. \tag{4}$$

From Equation (1), we have

$$\theta_1 + \theta_2 = 2(n-1)\binom{n-1}{k} \quad \text{and} \quad \theta_1\theta_2 = -\binom{n}{k}\binom{n}{k+1}.$$

Therefore, we have

$$\begin{aligned}\theta_1^2+\theta_2^2+2\binom{n-1}{k}(\theta_1+\theta_2) &= (\theta_1+\theta_2)^2-2\theta_1\theta_2+2\binom{n-1}{k}(\theta_1+\theta_2)\\ &= 4(n-1)^2\binom{n-1}{k}^2+2\binom{n}{k}\binom{n}{k+1}+4(n-1)\binom{n-1}{k}^2\\ &= 4n(n-1)\binom{n-1}{k}^2+2\binom{n}{k}\binom{n}{k+1}.\end{aligned}$$

It follows that

$$2\binom{n-1}{k}\theta_2+\theta_2^2=4n(n-1)\binom{n-1}{k}^2+2\binom{n}{k}\binom{n}{k+1}-\theta_1^2-2\binom{n-1}{k}\theta_1. \quad (5)$$

From (4) and (5), we have $m_1 = 1$ and thus $m_2 = n-1$ due to the first equation of (3) and $\theta_1+\theta_2 = 2(n-1)\binom{n-1}{k}$. Moreover, the multiplicity of 0 is $\zeta = |V(B(n,k))| - 1 - m_1 - m_2 = \binom{n}{k} + \binom{n}{k+1} - 1 - 1 - (n-1) = \binom{n+1}{k+1} - (n+1)$.

This completes the proof. ■

5. Conclusion

In this paper, we give the distance spectrum of $B(n,k)$ by showing that all its distance eigenvalues are contained in a quotient matrix. In fact, if Γ is graph having (distance) equitable partitions $\Pi_1, \Pi_2, \ldots, \Pi_l$ which share the same quotient matrix $B_{m\times m}$, then, for a (distance) eigenvalue of Γ not contained in those of $B_{m\times m}$, the corresponding eigenvector x sums to zero on each cell of Π_i for $1 \le i \le l$. Thus, we may get $l \times m$ functions. By considering the eigenfunction, we may get some additional functions. If l is large enough, then the system of functions may have no solution and thus all (distance) eigenvalues of Γ are contained in the eigenvalues of B. Applying this idea, one may get the (distance) spectra of graphs with high symmetry. We will try to obtain a precise condition for a graph admitting a (distance) partition whose quotient matrix has all eigenvalues of this graph in the future research.

Acknowledgments

The authors would like to thank the editor and anonymous referee for valuable suggestions, which have considerably improved the presentation of the paper.

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

This work is supported by the National Natural Science Foundation of China (NSFC) [Grant numbers 11531011, 11671344].

References

[1] Brouwer AE, Haemers WH. Spectra of graphs. Heidelberg: Springer; 2012.
[2] Aalipour G, Abiad A, Berikkyzy Z, et al. On the distance spectra of graphs. Linear Algebra Appl. 2016;497:66–87.
[3] Aouchiche M, Hansen P. Distance spectra of graphs: a survey. Linear Algebra Appl. 2014;458:301–386.
[4] Lin H, Hong Y, Wang J, et al. On the distance spectrum of graphs. Linear Algebra Appl. 2013;439:1662–1669.
[5] Koolen JH, Hayat S, Iqbal Q. Hypercubes are determined by their distance spectra. Linear Algebra Appl. 2016;505:97–108.
[6] Mirafzal SM. A new class of integral graphs constructed from the hypercube. Linear Algebra Appl. 2018;558:186–194.
[7] Huang XY, Huang QX. Note on the spectra of a class of graphs derived from set inclusion relations. arXiv: 1809.00889.
[8] Godsil CD, Royle G. Algebraic graph theory. Berlin: Springer-Verlag; 2001.
[9] Cvetković D, Rowlinson P, Simić S. An introduction to the theory of graph spectra. New York (NY): Cambridge University Press; 2010.
[10] Atik F, Panigrahi P. On the distance spectrum of distance regular graphs. Linear Algebra Appl. 2015;478:256–273.
[11] Lu L, Huang Q, Huang X. The graphs with exactly two distance eigenvalues different from -1 and -3. J Algebra Combin. 2017;45:629–647.