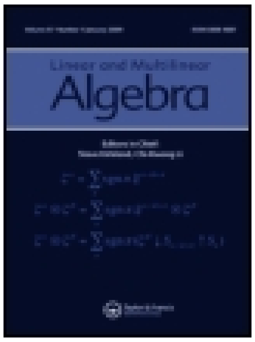

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On 2-equitable graphs

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ABSTRACT

Let Π be an equitable partition of a graph G with k cells. If G has no equitable partition with fewer cells than Π, then Π is called a minimal partition of G and G is called a k-equitable graph. It is clear that 1-equitable graphs are regular graphs. In this paper, we investigate 2-equitable graphs with four distinct eigenvalues.

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1. Introduction

Throughout this paper, we only consider undirected simple graphs. Let G be a graph on n vertices with vertex set $V = \{v_1, v_2, \dots, v_n\}$ and edge set E. The *adjacency matrix* of G is $A(G) = (a_{ij})_{n\times n}$ where $a_{ij} = 1$ if $v_i \sim v_j$ and 0 otherwise. Since A is symmetric, all eigenvalues of A are real and listed as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Such eigenvalues are called the *eigenvalues* of the graph G. If all distinct eigenvalues of G are $\lambda_1 > \lambda_2 > \cdots > \lambda_s$ and the multiplicity of λ_i is m_i, then the collection of all eigenvalues together with their multiplicities is the *spectrum* of G, denoted by $\mathrm{Sp}(G)$, and we always write $\mathrm{Sp}(G) = \{[\lambda_1]^{m_1}, [\lambda_2]^{m_2}, \dots, [\lambda_s]^{m_s}\}$. For $v \in V$, the *neighborhood* of v is $N_G(v) = \{u \in V \mid u \sim v\}$ and the *closed neighbourhood* of V is $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of v is $d_G(v) = |N_G(v)|$. If it is clear which graph we mean, we delete the subscript G in the notations like $d_G(v)$. For a subset $X \subseteq V$, the subgraph of G induced by X is denoted by $G[X]$. The *characteristic vector* $\delta_X \in \mathbb{R}^n$ of $X \subseteq V$ is a vector indexed by the vertices of V such that $\delta_X(v) = 1$ if $v \in X$ and 0 otherwise. If $X = \{v\}$, we always write δ_v for $\delta_{\{v\}}$.

As usual, we always write K_n, P_n and $K_{n_1,n_2,\dots,n_k}$ for the complete graph, the path and the complete multipartite graph on corresponding vertices. The *complement* $\overline{G}$ of a graph G is the graph with vertex set $V(\overline{G}) = V(G)$ and $uv \in E(\overline{G})$ if $uv \notin E(G)$. The *union* $G \cup H$ of two graphs G and H are the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. The *join* $G \nabla H$ of G and H is the graph obtained from $G \cup H$ by adding all edges between

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$V(G)$ and $V(H)$. Besides, we always write I_m for the unity matrix of order m, $J_{m\times n}$ the all-ones matrix of size $m \times n$ and j_n the all-ones vector of length n. For a positive integer t, let $e_1^{(t)}, e_2^{(t)}, \dots, e_t^{(t)}$ be the normal basis of $\mathbb{R}^t$, that is, $e_i^{(t)}$ is the vector such that $e_i^{(t)}(j) = 1$ if $j = i$ and 0 otherwise. Denote by $e_{ij}^{(t)} = e_i^{(t)} - e_j^{(t)}$ for $1 \le i, j \le t$. All other notations and marks not mentioned here are standard in [1].

Connected graphs with few distinct eigenvalues have aroused a lot of interest in the past decades. The problem to characterize such graphs was proposed by Doob [2]. It is known that the connected graphs with two distinct eigenvalues are complete graphs. The connected graphs with three distinct eigenvalues are far from being characterized. In 1965, Shrikhande and Bhagwandas [3] gave the classical result that a connected regular graph with three distinct eigenvalues is a strongly regular graph (a strongly regular graph with parameter (n, r, a, c) is an r-regular graph on n vertices such that $|N(u) \cap N(v)| = a$ if $u \sim v$ and $|N(u) \cap N(v)| = c$ if $u \not\sim v$). However, there are not many effective methods to construct strongly regular graphs. Even the existence of a strongly regular graph with some given parameters is still unknown. Therefore, many mathematicians turn their eyes to non-regular graphs with three distinct eigenvalues. Particularly, the connected non-regular graphs with three distinct eigenvalues having -2 as the smallest eigenvalue were completely determined by Dam [4]. With respect to graphs with four distinct eigenvalues, there are not many results. The connected regular bipartite graphs with four distinct eigenvalues are the incidence graphs of symmetric balanced incomplete block designs (SBIBD) [5]. Recently, Cioabă et al. in [6] (resp. [7]) determined the graphs with all but two eigenvalues different from ± 1 (resp. -2 and 0). Huang and Huang [8] determined the connected regular graphs with four distinct eigenvalues, in which two of them are simple, and having -1 as an eigenvalue. For more results on graphs with few distinct eigenvalues, we refer the reader to [9–18]. Motivated by the fact that there are few results about non-regular graphs with four distinct eigenvalues, we will investigate this problem.

In this paper, we give the definitions of k-equitable graphs and good partitions. We will see that 1-equitable graphs are regular graphs. Therefore, this notion can be regarded as a generalization of regular graphs. We investigate the 2-equitable graphs with three and four distinct eigenvalues, respectively. In detail, we completely determine the 2-equitable graphs with three distinct eigenvalues when they have a good partition. We also investigate the structure of 2-equitable graphs with four distinct eigenvalues when they have a good partition. Furthermore, we completely determine the 2-equitable graphs with four distinct eigenvalues under some spectral conditions. Lastly, we propose some conjectures to investigate in the future.

2. Definitions and observations

In this part, we first introduce some definitions and next give the spectra of some special graphs.

Let G be a connected graph with vertex set $V = \{v_1, v_2, \dots, v_n\}$ and adjacency matrix $A(G)$. A partition Π: $V = V_1 \cup V_2 \cup \cdots \cup V_m$ is called an *equitable partition* of G if, for $1 \le i, j \le m$ and $v_i \in V_i$, the number $|N(v_i) \cap V_j| = b_{ij}$ is a constant only depending on i and j but not the choice of v_i. The matrix $B_\Pi = (b_{i,j})_{m\times m}$ is called the quotient matrix of G with respect to Π. The matrix $P = [\delta_{V_1} \mid \delta_{V_2} \mid \cdots \mid \delta_{V_m}]$ is the *characteristic matrix* with respect to Π, where δ_{V_i} is the characteristic vector of V_i. It is easy to verify that $A(G)P = PB_\Pi$. From this equation, the following result follows.

Theorem 2.1 ([19, Page 198]): *Let G be a connected graph and Π an equitable partition of G with quotient matrix B_Π and characteristic matrix P. Then the eigenvalues of B_Π are also eigenvalues of $A(G)$. Furthermore, $A(G)$ has the following two kinds of eigenvectors:*

(i) *the eigenvectors in the column space of P, and the corresponding eigenvalues coincide with the eigenvalues of B_Π;*
(ii) *the eigenvectors orthogonal to the columns of P, i.e. those eigenvectors sum to zero on each cell of Π.*

Denote by $\Lambda_\Pi(G)$ the set of eigenvalues with an eigenvector of type (i) and denote by $\Lambda_{\overline{\Pi}}(G)$ the set of eigenvalues with an eigenvector of type (ii) of Theorem 2.1. Therefore, the set of distinct eigenvalues of G is the union of $\Lambda_\Pi(G)$ and $\Lambda_{\overline{\Pi}}(G)$. Note that the set $\Lambda_\Pi(G) \cap \Lambda_{\overline{\Pi}}(G)$ may be not empty. If $\Lambda_\Pi(G) \cap \Lambda_{\overline{\Pi}}(G) = \emptyset$, then (G, Π) is called a *good pair*.

An equitable partition Π is called *minimal* if G has no equitable partition with fewer cells than Π. If G has a minimal equitable partition with k cells, then G is called a *k-equitable* graph. If Π is a minimal partition of G such that (G, Π) is a good pair, then Π is called a *good partition*.

It is clear that a connected graph G is 1-equitable if and only if it is regular. The only minimal partition of a regular graph G is Π: $V(G) = V(G)$. It is known that the 1-equitable graphs with exactly three distinct eigenvalues are strongly regular graphs. In what follows, we would like to investigate the 2-equitable graphs with few distinct eigenvalues. By noticing that the connected graphs with exactly two distinct eigenvalues are complete graphs, we see that a connected 2-equitable graph has at least three distinct eigenvalues. We first characterize the 2-equitable graphs with exactly three distinct eigenvalues.

Theorem 2.2: *Assume that G is a connected 2-equitable graph with a good partition Π: $V = V_1 \cup V_2$. Then the graph G has exactly three distinct eigenvalues if and only if G is complete bipartite.*

Proof: The sufficiency is clear and we prove the necessity in what follows. Let $\mathcal{V} = \{x \in \mathbb{R}^{|G|} \mid x^T\delta_{V_1} = x^T\delta_{V_2} = 0\}$. Since G has three distinct eigenvalues and (G, Π) is a good pair, there is exactly one eigenvalue $\lambda \in \Lambda_{\overline{\Pi}}(G)$. Therefore, the eigenspace $\mathcal{E}(\lambda) = \mathcal{V}$. Suppose that there exist $u, v \in V_i$ such that $u \sim v$. Since $\delta_u - \delta_v \in \mathcal{V} = \mathcal{E}(\lambda)$, we have $\lambda(\delta_u - \delta_v) = A(\delta_u - \delta_v)$. By considering the u-th entries of both sides of this equation, we have $\lambda = -1$. Assume that $\Lambda_\Pi(G) = \{\theta_1, \theta_2\}$ with $\theta_1 > \theta_2$. We have $\theta_1 > \theta_2 > -1$ or $\theta_1 > -1 > \theta_2$. Note that the complete graph is the only connected graph with the smallest eigenvalue being -1 and is the only graph with just one non-negative eigenvalue, which is only feasible for the complete graph. However, G cannot be the complete graph since its spectrum does not satisfy inequalities $\theta_1 > \theta_2 > -1$ or $\theta_1 > -1 > \theta_2$. Thus, there is no edge in V_i for $1 \le i \le 2$, and thus G is bipartite. Note that the diameter of G is 2 due to the fact that it has three distinct eigenvalues. Thus G is complete bipartite. ■

Let $G = (V_1, V_2)$ be a bipartite graph and let t_1, t_2 be two positive integers. Denote by $G[t_1, t_2]$ the graph obtained from G by replacing each vertex $v \in V_i$ with a copy of K_{t_i} labelled as $K_{t_i}^v$ for $1 \le i \le 2$, and adding all edges between $K_{t_1}^u$ and $K_{t_2}^v$ if $u \sim v$ in G. A bipartite graph $H = (V_1, V_2)$ is called a (k_1, k_2)-semi-regular bipartite graph if each vertex in V_i has degree k_i for $1 \le i \le 2$. It is clear that Π: $V(H) = V_1 \cup V_2$ is an equitable partition of H (may be not minimal). Let $V_1' = \cup_{u \in V_1} V(K_{t_1}^u)$ and $V_2' = \cup_{v \in V_2} V(K_{t_2}^v)$. Obviously, the partition Π': $V(H[t_1, t_2]) = V_1' \cup V_2'$ is also an equitable partition of $H[t_1, t_2]$. The partition Π is called the *normal partition* of H and the partition Π' is called the *corresponding normal partition* of $H[t_1, t_2]$.

Lemma 2.1: *Let $H = (V_1, V_2)$ be a (k_1, k_2)-semi-regular bipartite graph with $|V_1| = m_1$ and $|V_2| = m_2$. For two positive integers t_1 and t_2, the spectrum of $H[t_1, t_2]$ is*

$$\left\{ [-1]^{m_1(t_1-1)+m_2(t_2-1)}, [t_1 - 1]^{\eta(B^T)}, [t_2 - 1]^{\eta(B)}, \frac{t_1 + t_2 - 2 \pm \sqrt{(t_1 - t_2)^2 + 4\lambda^2 t_1 t_2}}{2} \right\}$$

where λ are all positive eigenvalues of H, $\eta(B)$ and $\eta(B^T)$ are the nullities of B and B^T, and B is the incidence matrix from V_1 to V_2.

Proof: By assumption, $A(H) = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}$. Therefore, the adjacency matrix of $G = H[t_1, t_2]$ is given by

$$A(G) = \begin{pmatrix} I_{m_1} \otimes (J_{t_1 \times t_1} - I_{t_1}) & B \otimes J_{t_1 \times t_2} \\ B^T \otimes J_{t_2 \times t_1} & I_{m_2} \otimes (J_{t_2 \times t_2} - I_{t_2}) \end{pmatrix},$$

where $\otimes$ denotes the Kronecker product of two matrices. For $1 \le i \le m_1, 2 \le j \le t_1, 1 \le k \le m_2$ and $2 \le l \le t_2$, let $x_{ij} = \begin{pmatrix} e_i^{(m_1)} \otimes e_{1j}^{(t_1)} \\ 0 \end{pmatrix}$ and $y_{kl} = \begin{pmatrix} 0 \\ e_k^{(m_2)} \otimes e_{1l}^{(t_2)} \end{pmatrix}$. It is easy to verify that $A(G)x_{ij} = -x_{ij}$ and $A(G)y_{kl} = -y_{kl}$. Note that the vectors x_{ij} and y_{kl} are all linear independent. It means that -1 is an eigenvalue of G with multiplicity at least $m_1(t_1 - 1) + m_2(t_2 - 1)$.

It is known that the multiplicity of 0 as an eigenvalue of H is equal to $\eta(B) + \eta(B^T)$ where $\eta(B)$ and $\eta(B^T)$ are the nullities of B and B^T. Assume that $x_1, x_2, \ldots, x_s$ and $y_1, y_2, \ldots, y_t$ are the basis of the nullspaces of B^T and B respectively. Therefore, the vectors $\binom{x_1}{0}, \binom{x_2}{0}, \ldots, \binom{x_s}{0}$ and $\binom{0}{y_1}, \binom{0}{y_2}, \ldots, \binom{0}{y_t}$ form a set of orthogonal eigenvectors of H corresponding to 0. Let $X_i = \begin{pmatrix} x_i \otimes j_{t_1} \\ 0 \otimes j_{t_2} \end{pmatrix}$ and $Y_j = \begin{pmatrix} 0 \otimes j_{t_1} \\ y_j \otimes j_{t_2} \end{pmatrix}$ for $1 \le i \le s$ and $1 \le j \le t$. It is easy to verify that $A(G)X_i = (t_1 - 1)X_i$ and $A(G)Y_j = (t_2 - 1)Y_j$. Therefore, $t_1 - 1$ and $t_2 - 1$ are two eigenvalues of G with multiplicities at least $\eta(B^T)$ and $\eta(B)$ respectively.

Assume that $\begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} \binom{x}{y} = \lambda \binom{x}{y}$ where $\lambda > 0$. Let $\epsilon^+ = ((t_2 - t_1) + \sqrt{(t_1 - t_2)^2 + 4\lambda^2 t_1 t_2})/2\lambda t_2$ and $\epsilon^- = ((t_2 - t_1) - \sqrt{(t_1 - t_2)^2 + 4\lambda^2 t_1 t_2})/2\lambda t_2$. Let $X^+ = \begin{pmatrix} x \otimes j_{t_1} \\ \epsilon^+ y \otimes j_{t_2} \end{pmatrix}$ and $X^- = \begin{pmatrix} x \otimes j_{t_1} \\ \epsilon^- y \otimes j_{t_2} \end{pmatrix}$. One can verify that $A(G)X^+ = (t_1 + t_2 - 2 + \sqrt{(t_1 - t_2)^2 + 4\lambda^2 t_1 t_2})/2X^+$ and $A(G)X^- = (t_1 + t_2 - 2 - \sqrt{(t_1 - t_2)^2 + 4\lambda^2 t_1 t_2})/2X^-$. If H has p positive eigenvalues, then G has $2p$ such eigenvalues. Note that $2p + \eta(B) + \eta(B^t) = m_1 + m_2$. Hence, $2p + \eta(B) + \eta(B^T) + m_1(t_1 - 1) + m_2(t_2 - 1) = m_1 t_1 + m_2 t_2$. It means that we get all eigenvalues of G.

This completes the proof. ■

Remark 2.1: From the proof of Lemma 2.1, if the eigenvalue $\lambda > 0$ belongs to $\Lambda_{\overline{\Pi}}(H)$ then $(t_1 + t_2 - 2 \pm \sqrt{(t_1 - t_2)^2 + 4\lambda^2 t_1 t_2})/2$ belong to $\Lambda_{\overline{\Pi'}}(G)$, where Π is the normal partition of H and Π' is the corresponding normal partition of G.

Recall that a *balanced incomplete block design*, denoted by BIBD, consists of v elements and b subsets of these elements called blocks such that each element is contained in r blocks, each block contains k elements, and each pair of elements is simultaneously contained in λ blocks (see [20]). The integers (v, b, r, k, λ) are called the parameters of the design. In the case $r = k$ (and then $v = b$) the design is called *symmetric* with parameters (v, k, λ), denoted by (v, k, λ)-SBIBD.

The incidence graph of a BIBD is the bipartite graph on $b + v$ vertices (correspond to the blocks and elements of the design) with two vertices adjacent if and only if one corresponds to a block and the other corresponds to an element contained in that block. By simple observations, we get the following result.

Lemma 2.2: *Let $H = (V_1, V_2)$ be a (k_1, k_2)-semi-regular bipartite graph with $|V_1| = m_1$ and $|V_2| = m_2$. If there exist c_1 and c_2 such that $|N(u_i) \cap N(v_i)| = c_i$ for any $u_i, v_i \in V_i$ and $1 \le i \le 2$, then $m_1 = m_2 = m$, $k_1 = k_2 = k$, $c_1 = c_2 = c$ and H is the incidence graph of a (m, k, c)-SBIBD.*

Proof: Assume that B is the incidence matrix from V_1 to V_2. By the assumption, we have $BB^T = k_1 I + c_1(J - I)$ and $B^T B = k_2 I + c_2(J - I)$. Note that BB^T and $B^T B$ share the same non-zero eigenvalues. We have $k_1 + c_1(m_1 - 1) = k_2 + c_2(m_2 - 1)$ and $k_1 - c_1 = k_2 - c_2$. Also noticing that $k_1 m_1 = k_2 m_2$, we obtain that $m_1 = m_2 = m$, $k_1 = k_2 = k$ and $c_1 = c_2 = c$. The result follows. ■

Remark 2.2: It is clear that the spectrum of the incidence graph $H = (V_1, V_2)$ of a (m, k, c)-SBIBD is$\{\pm\sqrt{k + c(m-1)}, [\pm\sqrt{k-c}]^{m-1}\}$. Let Π: $V(H) = V_1 \cup V_2$ be the normal partition of H. It is clear that $\pm\sqrt{k + c(m-1)} \in \Lambda_{\Pi}(H)$ and $\pm\sqrt{k-c} \in \Lambda_{\overline{\Pi}}(H)$, and thus Π is a good partition. Combining Lemma 2.1 and Remark 2.1, the spectrum of $G = H[t_1, t_2]$ is given by

$$\left\{ [-1]^{m(t_1+t_2-2)}, \frac{t_1 + t_2 - 2 \pm \sqrt{(t_1 - t_2)^2 + 4(k + c(m-1))t_1 t_2}}{2}, \left[\frac{t_1 + t_2 - 2 \pm \sqrt{(t_1 - t_2)^2 + 4(k - c)t_1 t_2}}{2} \right]^{m-1} \right\}.$$

Let Π' be the corresponding normal partition of G. It is clear that

$$\Lambda_{\Pi'}(G) = \left\{ \frac{t_1 + t_2 - 2 \pm \sqrt{(t_1 - t_2)^2 + 4(k + c(m-1))t_1 t_2}}{2} \right\}$$

and all other eigenvalues belong to $\Lambda_{\overline{\Pi'}}(G)$, and thus Π' is also a good partition.

3. Main results

Let $\mathcal{G}_2^n$ be the set of all connected 2-equitable graphs on n vertices and $\mathcal{H}_2^n$ the subset of 2-equitable graphs possessing a good partition. Define

$$\begin{aligned}
\mathcal{H}_2^n(4) &= \{G \in \mathcal{H}_2^n \mid |\Lambda_\Pi(G)| + |\Lambda_{\overline{\Pi}}(G)| = 4 \text{ for a good partition } \Pi\},\\
\mathcal{H}_2^n(4,0) &= \{G \in \mathcal{H}_2^n(4) \mid 0 \in \Lambda_{\overline{\Pi}}(G) \text{ for a good partition } \Pi\},\\
\mathcal{H}_2^n(4,-1) &= \{G \in \mathcal{H}_2^n(4) \mid -1 \in \Lambda_{\overline{\Pi}}(G) \text{ for a good partition } \Pi\},\\
\mathcal{H}_2^n(4,-1,0) &= \{G \in \mathcal{H}_2^n(4) \mid \Lambda_{\overline{\Pi}}(G) = \{-1,0\} \text{ for a good partition } \Pi\},\\
\mathcal{H}_2^n(4,-) &= \{G \in \mathcal{H}_2^n(4) \mid -1, 0 \notin \Lambda_{\overline{\Pi}}(G) \text{ for a good partition } \Pi\}.
\end{aligned}$$

It is clear that $\mathcal{H}_2^n(4) = \mathcal{H}_2^n(4,0) \cup \mathcal{H}_2^n(4,-1) \cup \mathcal{H}_2^n(4,-)$ and $\mathcal{H}_2^n(4,0) \cap \mathcal{H}_2^n(4,-1) = \mathcal{H}_2^n(4,-1,0)$. Our goal is to characterize $\mathcal{H}_2^n(4)$. In this part, we completely determine $\mathcal{H}_2^n(4,0)$ and $\mathcal{H}_2^n(4,-1)$, and give a structural description for all graphs in $\mathcal{H}_2^n(4)$.

Lemma 3.1: *If $G \in \mathcal{H}_2^n(4,0)$ is a graph with good partition Π: $V = V_1 \cup V_2$, then $N(u) = N(v)$ for any $u, v \in V_i$ with $u \not\sim v$ and $1 \le i \le 2$.*

Proof: Assume that $\Lambda_\Pi(G) = \{\theta_1, \theta_2\}$ and $\Lambda_{\overline{\Pi}}(G) = \{0, \lambda\}$. Let $\mathcal{V} = \{x \in \mathbb{R}^n \mid x^T\delta_{V_1} = x^T\delta_{V_2} = 0\}$ where $n = |G|$. Obviously, $\mathcal{V} = \mathcal{E}(0) \oplus \mathcal{E}(\lambda)$. For any pair of vertices $u, v \in V_i$ with $u \not\sim v$, since $\delta_u - \delta_v \in \mathcal{V} = \mathcal{E}(0) \oplus \mathcal{E}(\lambda)$, we may assume that $\delta_u - \delta_v = ax + by$ where $x \in \mathcal{E}(0) \setminus \{0\}$ and $y \in \mathcal{E}(\lambda) \setminus \{0\}$. If $b \ne 0$ then $A(\delta_u - \delta_v) = A(ax + by) = b\lambda y$. Therefore, $\alpha = A(\delta_u - \delta_v) \in \mathcal{E}(\lambda) \setminus \{0\}$. Note that $\alpha(i) = 1$ if $i \in N(u) \setminus N(v)$, $\alpha(i) = -1$ if $i \in N(v) \setminus N(u)$ and $\alpha(i) = 0$ otherwise. By successively considering the u-th entries and the v-th entries of both sides of $A\alpha = \lambda\alpha$, we have $|N(u) \setminus N(v)| = |N(v) \setminus N(u)| = 0$. It leads to that $N(u) = N(v)$. Therefore, $\alpha = A(\delta_u - \delta_v) = 0$, a contradiction. Thus, we have $b = 0$ and $\delta_u - \delta_v \in \mathcal{E}(0) \setminus \{0\}$. It means that $A(\delta_u - \delta_v) = 0$, which leads to $N(u) = N(v)$. ■

Lemma 3.2: *If $G \in \mathcal{H}_2^n(4,-1)$ is a graph with good partition Π: $V = V_1 \cup V_2$, then $N[u] = N[v]$ for any $u, v \in V_i$ with $u \sim v$ and $1 \le i \le 2$.*

Proof: Assume that $\Lambda_\Pi(G) = \{\theta_1, \theta_2\}$ and $\Lambda_{\overline{\Pi}}(G) = \{-1, \lambda\}$. Let $\mathcal{V} = \{x \in \mathbb{R}^n \mid x^T\delta_{V_1} = x^T\delta_{V_2} = 0\}$ where $n = |G|$. Obviously, $\mathcal{V} = \mathcal{E}(-1) \oplus \mathcal{E}(\lambda)$. For any pair of vertices $u, v \in V_i$ with $u \sim v$, since $\delta_u - \delta_v \in \mathcal{V} = \mathcal{E}(-1) \oplus \mathcal{E}(\lambda)$, we may assume that $\delta_u - \delta_v = ax + by$ where $x \in \mathcal{E}(-1) \setminus \{0\}$ and $y \in \mathcal{E}(\lambda) \setminus \{0\}$. If $b \ne 0$ then $(A+I)(\delta_u - \delta_v) = (A+I)(ax + by) = b(\lambda + 1)y$. Therefore, $\alpha = (A+I)(\delta_u - \delta_v) \in \mathcal{E}(\lambda) \setminus \{0\}$. Note that $\alpha(i) = 1$ if $i \in N[u] \setminus N[v]$, $\alpha(i) = -1$ if $i \in N[v] \setminus N[u]$ and $\alpha(i) = 0$ otherwise. By successively considering the u-th entries and the v-th entries of both sides of $A\alpha = \lambda\alpha$, we have $|N[u] \setminus N[v]| = |N[v \setminus N[u]| = 0$. It leads to that $N[u] = N[v]$. Therefore, $\alpha = A(\delta_u - \delta_v) = 0$, a contradiction. Thus, we have $b = 0$ and $\delta_u - \delta_v \in \mathcal{E}(-1) \setminus \{0\}$. It means that $A(\delta_u - \delta_v) = -(\delta_u - \delta_v)$ which leads to $N[u] = N[v]$. ■

From Lemmas 3.1 and 3.2, we get the following result.

Lemma 3.3: *The graph set*

$$\mathcal{H}_2^n(4,-1,0) = \{K_s \nabla \overline{K_t} \mid s,t \geq 2\}.$$

Proof: Since $\Lambda_{\overline{\Pi}}(G) = \{0,-1\}$, Lemmas 3.1 and 3.2 indicate that each pair of vertices in the same cell share the same neighbourhood. Therefore, $G[V_i]$ is either a complete graph or an empty graph and $G = G[V_1]\nabla G[V_2]$. Clearly, $G[V_1]$ and $G[V_2]$ cannot be both complete or both empty. Therefore, $G = K_s\nabla\overline{K_t}$ and $s,t \geq 2$. Note that $K_s\nabla\overline{K_t} = H[s,1]$ where $H = K_{1,t}$. Since $\mathrm{Sp}(H) = \{\pm\sqrt{t}, [0]^{t-1}\}$, $\eta(B^T) = 0$ and $\eta(B) = t-1$ where $B = j_t^T$, Lemma 2.1 implies that

$$\mathrm{Sp}(K_s\nabla\overline{K_t}) = \left\{\left(s-1 \pm \sqrt{4st + (s-1)^2}\right)/2, [0]^{t-1}, [-1]^{s-1}\right\}.$$

The result follows. ■

Lemma 3.4: *Let $G \in \mathcal{H}_2^n(4,0)$ be a graph with good partition Π: $V = V_1 \cup V_2$. If $\Lambda_{\overline{\Pi}}(G) = \{0,\lambda\}$ with $\lambda \neq -1$ then $G = \overline{H[t_1,t_2]}$ where H is the incidence graph of a SBIBD and $t_1 \neq t_2$.*

Proof: Since $0 \in \Lambda_{\overline{\Pi}}(G)$, Lemma 3.1 indicates that each pair of non-adjacent vertices share the same neighbourhood. It means that $G[V_i]$ contains no induced $K_2 \cup K_1$ for $1 \leq i \leq 2$. Therefore, $G[V_i]$ are complete multipartite graphs. Moreover, they are balanced complete multipartite graphs due to that $G[V_i]$ are regular. Assume $G[V_1] = \overline{s_1K_{t_1}}$ and $G[V_2] = \overline{s_2K_{t_2}}$.

Assume that $\Lambda_{\overline{\Pi}}(G) = \{0,\lambda\}$ with $\lambda \neq -1$. Let $\mathcal{V} = \{x \in \mathbb{R}^n \mid x^T\delta_{V_1} = x^T\delta_{V_2} = 0\}$ where $n = |G|$. Obviously, $\mathcal{V} = \mathcal{E}(0) \oplus \mathcal{E}(\lambda)$. For any pair of vertices $u,v \in V_i$ with $u \sim v$, since $\delta_u - \delta_v \in \mathcal{V} = \mathcal{E}(0) \oplus \mathcal{E}(\lambda)$, assume that $\delta_u - \delta_v = ax + by$ where $x \in \mathcal{E}(0) \setminus \{0\}$ and $y \in \mathcal{E}(\lambda) \setminus \{0\}$. If $b = 0$ then $\delta_u - \delta_v \in \mathcal{E}(0)$. Therefore, by considering the u-th entries of both sides of $A(\delta_u - \delta_v) = 0$, we have $-1 = 0$, a contradiction. Thus, $b \neq 0$ and $\alpha = A(\delta_u - \delta_v) = A(ax+by) = b\lambda y \in \mathcal{E}(\lambda)$. Note that $\alpha(i) = 1$ if $i \in N(u) \setminus N(v)$, $\alpha(i) = -1$ if $i \in N(v) \setminus N(u)$ and $\alpha(i) = 0$ otherwise. By considering the u-th entries and the v-th entries of $A\alpha = \lambda\alpha$, we have $|N(u) \setminus N(v)| = |N(v) \setminus N(u)| = -\lambda$. Therefore, there are totally $n - d_i + \lambda$ vertices adjacent to neither u nor v, where d_i is the valency of vertices in V_i. Moreover, since $G[V_i]$ is complete multipartite, there is no $w \in V_i$ such that $w \not\sim u$ and $w \not\sim v$. Therefore, all such $n - d_i + \lambda$ vertices belong to V_j where $j \in \{1,2\} \setminus \{i\}$.

Now we consider $\overline{G}$. By arguments above, $\overline{G}[V_i] = s_iK_{t_i}$. Assume that $U_1, U_2, \ldots, U_{s_i} \subset V_1$ and $U_1', U_2', \ldots, U_{s_2}' \in V_2$ such that $\overline{G}[U_k] = K_{t_1}$ and $\overline{G}[U_l'] = K_{t_2}$ for $1 \leq k \leq s_1$ and $1 \leq l \leq s_2$. Since for any two vertices $u,v \in U_k$, they are not adjacent in G. Lemma 3.1 indicates that $N_G(u) = N_G(v)$ and thus $N_{\overline{G}}[u] = N_{\overline{G}}[v]$. It leads to that each vertex in U_k is adjacent to every vertex in U_l' if there is an edge between U_k and U_l'. Thus, $\overline{G} = H[t_1,t_2]$ where $H = (U,V)$ is a semi-regular bipartite graph. For any $u_1,u_2 \in U$, they can be viewed as two vertices in, say, U_1 and U_2, and they are adjacent in G. Therefore, there are $n - d_1 + \lambda$ vertices in V_2 adjacent to neither u_1 nor u_2. It means that $|N_{\overline{G}}(u_1) \cap N_{\overline{G}}(u_2)| = n - d_1 + \lambda$. Therefore, $|N_H(u_1) \cap N_H(u_2)| = (n - d_1 + \lambda)/t_2 = c_1$. Similarly, for any $v_1, v_2 \in V$, we have $|N_H(v_1) \cap N_H(v_2)| = (n - d_2 + \lambda)/t_1 = c_2$. Lemma 2.2 implies that H is the incidence graph of a SBIBD.

This completes the proof. ■

We present the following result without proving since the proof is very similar to that of Lemma 3.4.

Lemma 3.5: *Let $G \in \mathcal{H}_2^n(4,-1)$ be a graph with good partition Π: $V = V_1 \cup V_2$. If $\Lambda_{\overline{\Pi}}(G) = \{-1, \lambda\}$ with $\lambda \neq 0$ then $G = H[t_1, t_2]$, where H is the incidence graph of a SBIBD and $t_1 \neq t_2$.*

Combining Lemmas 3.3, 3.4 and Remark 2.2, we get the following result.

Theorem 3.1: *The graph set*

$$\mathcal{H}_2^n(4,0) = \{K_s \nabla \overline{K_t}, \overline{H[t_1,t_2]} \mid s,t \geq 2, H = \mathrm{GSB}(m,k,c) \text{ with } k-c=1 \text{ and } t_1 \neq t_2\},$$

where $\mathrm{GSB}(m,k,c)$ *denotes the incidence graph of an* (m,k,c)*-SBIBD.*

Proof: Assume that $G \in \mathcal{H}_2^n(4,0)$. Lemmas 3.3 and 3.4 indicate that $G = K_s \nabla \overline{K_t}$ or $\overline{H[t_1,t_2]}$. It suffices to show that $G = \overline{H[t_1,t_2]} \in \mathcal{H}_2^n(4,0)$ when $H = \mathrm{GSB}(m,k,c)$ with $k-c = 1$ and $t_1 \neq t_2$. Assume that Π is the corresponding normal partition of $\overline{G} = H[t_1,t_2]$, which is also an equitable partition of G. Note that $A(G) = J - I - A(H[t_1,t_2])$ and $x^T j_n = 0$ for any $x \in \mathcal{E}_{\overline{G}}(\lambda)$ and $\lambda \in \Lambda_{\overline{\Pi}}(\overline{G})$. Therefore, each eigenvalue $\lambda \in \Lambda_{\overline{\Pi}}(\overline{G})$ leads to an eigenvalue $-1-\lambda$ of G. Thus, from Remark 2.2, the spectrum of G is

$$\left\{\theta_1, \theta_2, \left[\frac{-(t_1+t_2) \pm \sqrt{(t_1-t_2)^2 + 4(k-c)t_1t_2}}{2}\right]^{m-1}, [0]^{m(t_1+t_2-2)}\right\},$$

where $\Lambda_\Pi(G) = \{\theta_1, \theta_2\}$ and other eigenvalues belong to $\Lambda_{\overline{\Pi}}(G)$. It means that Π is a good partition. Furthermore, from $k-c = 1$, we have $(-(t_1+t_2) + \sqrt{(t_1-t_2)^2 + 4(k-c)t_1t_2})/2 = 0$. Thus, $G = \overline{H[t_1,t_2]} \in \mathcal{H}_2^n(4,0)$. ■

Similarly, from Lemmas 3.3, 3.5 and Remark 2.2, we get the following result. We omit the proof since it is very similar to the proof of Theorem 3.1.

Theorem 3.2: *The graph set*

$$\mathcal{H}_2^n(4,-1) = \{K_s \nabla \overline{K_t}, H[t_1,t_2] \mid s,t \geq 2, H = \mathrm{GSB}(m,k,c) \text{ with } k-c=1 \text{ and } t_1 \neq t_2\},$$

where $\mathrm{GSB}(m,k,c)$ *denotes the incidence graph of a* (m,k,c)*-SBIBD.*

It remains to consider the graph set $\mathcal{H}_2^n(4,-)$.

Theorem 3.3: *If $G \in \mathcal{H}_2^n(4,-)$ is a graph with good partition Π: $V = V_1 \cup V_2$, then, for any $u, v \in V_i$, $|N(u) \cap N(v)| = \gamma_i$ when $u \sim v$ and $|N(u) \cap N(v)| = \mu_i$ when $u \not\sim v$. Furthermore,$\gamma_1 - \gamma_2 = \mu_1 - \mu_2 = d_1 - d_2$ where d_i is the valency of the vertices in V_i.*

Proof: Assume that $\Lambda_{\overline{\Pi}}(G) = \{\lambda_1, \lambda_2\}$ with $\lambda_1, \lambda_2 \notin \{0,-1\}$. Let $\mathcal{V} = \{x \in \mathbb{R}^n \mid x^T \delta_{V_1} = x^T \delta_{V_2} = 0\}$. It is clear that $\delta_u - \delta_v \in \mathcal{V}$ for any pair of vertices $u, v \in V_i$ and $1 \leq i \leq 2$.

For any $u, v \in V_i$ with $u \not\sim v$. Assume that $\delta_u - \delta_v = ax + by$ where $x \in \mathcal{E}(\lambda_1)$ and $y \in \mathcal{E}(\lambda_2)$. If one of a and b equals to 0, say $a = 0$, then $\delta_u - \delta_v \in \mathcal{E}(\lambda_2)$. By considering the

u-th entries and the v-th entries of $A(\delta_u - \delta_v) = \lambda_2(\delta_u - \delta_v)$, we have $N(u) = N(v)$ and thus $\lambda_2 = 0$, a contradiction. Thus, $a, b \neq 0$ and we may assume that $\delta_u - \delta_v = x + y$. From the two equations $\delta_u - \delta_v = x + y$ and $A(\delta_u - \delta_v) = A(x + y) = \lambda_1 x + \lambda_2 y$, we have

$$x(i) + y(i) = \begin{cases} 1, & i = u \\ -1, & i = v \\ 0, & \text{otherwise} \end{cases} \quad \text{and } \lambda_1 x(i) + \lambda_2 y(i) = \begin{cases} 1, & i \in N(u) \setminus N(v) \\ -1, & i \in N(v) \setminus N(u) \\ 0, & \text{otherwise} \end{cases}.$$

It leads to that

$$x(i) = \begin{cases} \dfrac{-\lambda_2}{\lambda_1 - \lambda_2}, & i = u \\ \dfrac{\lambda_2}{\lambda_1 - \lambda_2}, & i = v \\ \dfrac{1}{\lambda_1 - \lambda_2}, & i = N(u) \setminus N(v) \\ \dfrac{-1}{\lambda_1 - \lambda_2}, & i = N(v) \setminus N(u) \\ 0, & \text{otherwise} \end{cases} \quad \text{and } y(i) = \begin{cases} \dfrac{\lambda_1}{\lambda_1 - \lambda_2}, & i = u \\ \dfrac{-\lambda_1}{\lambda_1 - \lambda_2}, & i = v \\ \dfrac{-1}{\lambda_1 - \lambda_2}, & i = N(u) \setminus N(v) \\ \dfrac{1}{\lambda_1 - \lambda_2}, & i = N(v) \setminus N(u) \\ 0, & \text{otherwise} \end{cases}.$$

By successively considering the u-th entries and the v-th entries of both sides of $Ax = \lambda_1 x$, we have $|N(u) \setminus N(v)| = |N(v) \setminus N(u)| = -\lambda_1\lambda_2$. Note that $d(u) = d(v) = d_i$. We have $|N(u) \cap N(v)| = d_i + \lambda_1\lambda_2$.

Similarly, for any $u, v \in V_i$ with $u \sim v$, we have $|N(u) \cap N(v)| = d_i + (\lambda_1 + 1)(\lambda_2 + 1)$.

This completes the proof. ■

Let G be a graph with an equitable partition Π: $V_1 \cup V_2$ such that $|V_i| = n_i$ and each vertex in $G[V_i]$ has valency d_i. For $1 \leq i \leq 2$ and any pair of vertices $u, v \in V_i$, if $|N(u) \cap N(v)| = \gamma_i$ for $u \sim v$ and $|N(u) \cap N(v)| = \mu_i$ for $u \not\sim v$, then G is called $(n_1, n_2; d_1, d_2; \gamma_1, \gamma_2; \mu_1, \mu_2)$-*semi-strongly regular*. Theorem 3.3 indicates that, if $G \in \mathcal{H}_2^n(4, -)$ with a good partition Π such that $\Lambda_{\overline{\Pi}} = \{\lambda_1, \lambda_2\}$ then G is $(n_1, n_2; d_1, d_2; d_1 + (\lambda_1 + 1)(\lambda_2 + 1), d_2 + (\lambda_1 + 1)(\lambda_2 + 1); d_1 + \lambda_1\lambda_2, d_2 + \lambda_1\lambda_2)$-semi-strongly regular. Note that all graphs in $\mathcal{H}_2^n(4, 0)$ and $\mathcal{H}_2^n(4, -1)$ are also semi-strongly regular. We present the following result.

Theorem 3.4: *Let G be a 2-equitable graph with good partition Π: $V = V_1 \cup V_2$. If G has exactly four distinct eigenvalues then it is semi-strongly regular.*

From Theorem 3.4, we can easily get the following corollary.

Corollary 3.1: *Let G be a 2-equitable graph with a good partition Π: $V = V_1 \cup V_2$. Assume that $|V_1| = 1$. Then G has exactly four distinct eigenvalues if and only if $G = K_1 \nabla H$ where H is a strongly regular graph with parameter different from the form $(r^3 + 2r^2, r^2 + r, r, r)$.*

Proof: It is clear that $G = K_1 \nabla H$ for some H. Theorem 3.4 means that G is semi-strongly regular, which implies that H is strongly regular. Note that, if the parameter of H is of the form $(r^3 + 2r^2, r^2 + r, r, r)$, then G has three distinct eigenvalue [18, Lemma 4.1]. It follows the result. ■

4. Conclusion

In this paper, we introduce the definition of k-equitable graphs and investigate the 2-equitable graphs with three and four distinct eigenvalue. However, this definition relies on the so-called minimal equitable partition of a graph. Therefore, it is crucial to study the uniqueness of the minimal partition. Let Π: $V(G) = V_1 \cup V_2$ and Π': $V(G) = V_1' \cup V_2'$ be two equitable partitions of the graph G. If there is an automorphism σ of G such that $\sigma(V_1) = V_1'$ (or $\sigma(V_1) = V_2'$) and $\sigma(V_2) = V_2'$ (resp. $\sigma(V_2) = V_1'$), then Π and Π' are *isomorphic*.

Conjecture 4.1: If G is a 2-equitable graph then the minimal equitable partition of G is unique up to isomorphism.

This conjecture may be solved by investigating the structure of the graph.

Note that all our result are obtained under the condition that a graph has a good partition. It is believable that almost all graphs have a good partition.

Conjecture 4.2: Almost all 2-equitable graphs have a good partition, that is, $|\mathcal{H}_2^n|/|\mathcal{G}_2^n| \to 1$ along with $n \to \infty$.

We do not get any clue to solve this conjecture.

From Theorem 3.4, a graph $G \in \mathcal{H}_2^n(4)$ must be a semi-strongly regular graph. Assume that Π: $V(G) = V_1 \cup V_2$ is a good partition of G and the adjacency matrix of G is $A = \left(\begin{smallmatrix} X & B \\ B^T & Y \end{smallmatrix}\right)$ where B is the incidence matrix from V_1 to V_2. If $A' = \left(\begin{smallmatrix} 0 & B \\ B^T & 0 \end{smallmatrix}\right)$ is the incidence graph of a BIBD, then, for $u, v \in V_i$ and $j = \{1, 2\} \setminus \{i\}$, $|(N(u) \cap V_j) \cap (N(v) \cap V_j)|$ is a constant depending on whether u and v are adjacent or not. It means that $G[V_1]$ and $G[V_2]$ are strongly regular. Thus, one may construct graphs in $\mathcal{H}_2^n(4, -)$ from two strongly regular graphs H_1 and H_2 by adding edges between them such that the deletion of all edges in H_i leads to the incidence graph of a BIBD.

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References

[1] Cvetković D, Rowlinson P, Simić S. An introduction to the theory of graph spectra. New York: Cambrige University Press; 2010.

[2] Doob M. Graphs with a small number of distinct eigenvalues. Ann New York Acad Sci. 1970;175:104–110.

[3] Shrikhande SS, Bhagwandas. Duals of incomplete block designs. J Indian Stat Assoc. 1965;3:30–37.
[4] van Dam ER. Non regular graphs with three eigenvalues. J Comb Theory Ser B. 1998;73:101–118.
[5] Brouwer AE, Haemers WH. Spectra of graphs. NewYork: Springer; 2011.
[6] Cioabă SM, Haemers WH, Vermette JR, et al. The graphs with all but two eigenvalues equal to ± 1. J Algebraic Comb. 2015;41(3):887–897.
[7] Cioabă SM, Haemers WH, Vermette JR. The graphs with all but two eigenvalues equal to -2 or 0. Designs Codes Cryptography. 2017;84(1–2):153–163.
[8] Huang X, Huang Q. On regular graphs with four distinct eigenvalues. Linear Algebra Appl. 2017;512:219–233.
[9] Bridges WG, Mena RA. Multiplicative cones–a family of three eigenvalue graphs. Aequ Math. 1981;22:208–214.
[10] de Caen D, van Dam ER, Spence E. A nonregular analogue of conference graphs. J Comb Theory Ser A. 1999;88:194–204.
[11] Cheng XM, Gavrilyuk AL, Greaves GRW, et al. Biregular graphs with three eigenvalues. Europ J Comb. 2016;56:57–80.
[12] van Dam ER. Regular graphs with four eigenvalues. Linear Algebra Appl. 1995;226–228: 139–163.
[13] van Dam ER, Koolen JH, Xia ZJ. Graphs with many valencies and few eigenvalues. Electron J Linear Algebra. 2015;28:12–24.
[14] van Dam ER, Spence E. Small regular graphs with four eigenvalues. Discrete Math. 1998;189:233–257.
[15] van Dam ER, Spence E. Combinatorial designs with two singular values I: uniform multiplicative designs. J Comb Theory Ser A. 2004;107:127–142.
[16] van Dam ER, Spence E. Combinatorial designs with two singular values II. Partial geometric designs. Linear Algebra Appl. 2005;396:303–316.
[17] Muzychuk M, Klin M. On graphs with three eigenvalues. Discrete Math. 1998;189:191–207.
[18] Rowlinson P. On graphs with just three distinct eigenvalues. Linear Algebra Appl. 2016;507:462–473.
[19] Godsil CD, Royle G. Algebraic graph theory. Berlin: Springer-Verlag; 2001.
[20] Cvetković DM, Doob M, Sachs H. Spectra of graphs: theory and applications, V.E.B. Berlin: Deutscher Verlag der Wissenschaften; 1979.