J. Group Theory, Ahead of Print
DOI 10.1515/jgth-2023-0041

5-Regular prime graphs of finite nonsolvable groups

Qinghong Guo, Weijun Liu* and Lu Lu

Communicated by Hung Tong-Viet

Abstract. The prime graph $\Delta(G)$ of a finite group G is a graph whose vertex set is the set of prime factors of the degrees of all irreducible complex characters of G, and two distinct primes p and q are joined by an edge if the product pq divides some character degree of G. In 2014, Tong-Viet [H. P. Tong-Viet, Finite groups whose prime graphs are regular, *J. Algebra* **397** (2014), 18–31] proposed the following conjecture. Let G be a group and let $k \geq 5$ be odd. If the prime graph $\Delta(G)$ is k-regular, then $\Delta(G)$ is a complete graph of order $k+1$. In this paper, we show that if the prime graph $\Delta(G)$ of a finite nonsolvable group G is 5-regular, then $\Delta(G)$ is isomorphic to the complete graph K_6 or possibly the graph depicted in the first figure below. Moreover, if G is an almost simple group, then $\Delta(G)$ is isomorphic to the complete graph K_6.

1 Introduction

All groups considered in this paper are finite. The letter G always denotes a finite group. For an integer n, we write $\pi(n)$ for the set of all prime divisors of n. The order of G is denoted by $|G|$, and we write $\pi(G)$ instead of $\pi(|G|)$ for the set of all prime divisors of $|G|$. Let G^n denote the direct product of n copies of G. We denote by (m,n) the greatest common divisor of integers m and n. We refer to Atlas [4] for the notation of finite nonabelian simple groups. A group G is said to be an *almost simple group* with socle S if there exists a nonabelian simple group S such that $S \trianglelefteq G \leq \mathrm{Aut}(S)$.

All graphs considered in this paper are finite, simple and undirected. The *degree* of a vertex v in a simple undirected graph Δ, denoted by $\deg(v)$, is the number of neighbors of v in Δ. In a graph, a vertex is said to be a *complete vertex* if it is adjacent to all the other vertices. A graph is said to be a *complete graph* if every vertex is complete, and a complete graph of order n is denoted by K_n. A graph is called K_n*-free* if it has no subgraph isomorphic to K_n. Furthermore, a graph is called k*-regular* if every vertex has degree k for some nonnegative integer k. The

This research was supported by NSFC (Nos. 12071484 and 12271524), NSF of Hunan (No. 2022JJ30674) and the Foundation of Guangdong University of Science and Technology.

clique number $\omega(\Delta)$ of a graph Δ is the maximum size of a subset of $V(\Delta)$, the vertex set of Δ, inducing a complete subgraph.

All characters considered in this paper are complex characters. Let $\mathrm{Irr}(G)$ be the set of all irreducible complex characters of G, and $\mathrm{cd}(G)$ the set of all irreducible complex character degrees of G, that is, $\mathrm{cd}(G) = \{\chi(1) : \chi \in \mathrm{Irr}(G)\}$. We write $\rho(G)$ to denote the set of all primes which divide some character degree of G, that is, $\rho(G) = \{p : p$ is a prime and $p \mid a$ for some $a \in \mathrm{cd}(G)\}$. If λ is a character of G, then we write $\mathrm{Irr}(\lambda)$ for the set of all irreducible constituents of λ in G and $\mathrm{cd}(\lambda) = \{\chi(1) : \chi \in \mathrm{Irr}(\lambda)\}$. If N is a normal subgroup of G and $\theta \in \mathrm{Irr}(N)$, then the inertia group of θ in G is denoted by $I_G(\theta)$. Given a finite group G, the degree set $\mathrm{cd}(G)$ of its complex irreducible characters encodes a great deal of structural information about the group G. For example, the well-known result due to Thompson [9, Corollary 12.2] says that if a prime p divides every nonlinear character degree of a group G, then G has a normal p-complement. According to [9, Corollary 12.6 and Theorem 12.15], we deduce that if the number of degrees regardless of multiplicity of irreducible characters of G is at most 3, then G is solvable. In other words, if $|\mathrm{cd}(G)| \le 3$, then G is solvable.

The *character-prime graph* (*prime graph* for short) $\Delta(G)$ corresponding to a group G is a simple undirected graph whose vertex set is $\rho(G)$, and there is an edge between two distinct vertices p and q if and only if the product pq divides some character degree of G. This graph was first introduced by Manz, Staszewski and Willems [14] and has been studied extensively since then. Note that if N is a normal subgroup of G, then $\Delta(G/N)$ and $\Delta(N)$ are subgraphs of $\Delta(G)$. The study of prime graphs related to character degrees has its own interest. We will study the following problem.

Problem 1. Which graphs can occur as the prime graphs of finite groups?

This is one of the attractive topics in the character theory of finite groups. For example, Manz, Staszewski and Willems [14] proved that $\Delta(G)$ has at most three connected components. For finite solvable group, one of the most remarkable results is due to Pálfy [20]. In that paper, Pálfy proved that, for any solvable group G and choice of three distinct primes in $\rho(G)$, there exists an irreducible character of G with degree divisible by at least two of those primes. Unfortunately, this is generally not true for finite nonsolvable groups, e.g., $\mathrm{cd}(A_5) = \{1, 3, 4, 5\}$. Later, Moretó and Tiep [18] showed that, for an arbitrary finite group G and choice of four distinct primes in $\rho(G)$, Pálfy's conclusion holds, and this also generalized the conclusion in [14]. The two results are very useful in determining which graphs can occur as the prime graphs of finite solvable and nonsolvable groups, respectively. For example, the graph with four vertices and no edges cannot occur as the prime graph of a finite group. The prime graph $\Delta(G)$ has at most two connected components if G is solvable; furthermore, if $\Delta(G)$ has exactly two connected components, then each connected component is complete. Zhang [31] characterized the finite solvable groups with nonconnected prime graphs. Later on, Lewis [11] completely classified the finite solvable groups whose prime graphs have exactly two connected components. On the other hand, Huppert [7] classified all possible simple graphs with at most four vertices which can occur as the prime graph of some finite solvable group except possibly a path with four vertices. Zhang [30] proved that a path with four vertices cannot occur as the prime graph of a finite solvable group. Lewis and White [13] proved that the same conclusion holds for finite nonsolvable groups. In [24–26], White determined the prime graphs of all finite nonabelian simple groups of Lie type, and he summarized the prime graphs of sporadic simple groups and alternating groups in [27]. We will see that prime graphs of many finite nonabelian simple groups are complete. It seems that the prime graphs of finite groups contain many edges. An interesting special case of Problem 1 is to determine which regular graphs can occur as the prime graph of a finite group. In 2014, Tong-Viet [23] proved that the prime graph $\Delta(G)$ of a finite group G is 3-regular if and only if it is a complete graph with four vertices, that is, K_4. In that paper, Tong-Viet proposed the following conjecture.

Conjecture 1. *Let G be a group and $k \geq 2$ an integer. Suppose that the prime graph $\Delta(G)$ is k-regular. Then*

(1) *if $k \geq 5$ is odd, then $\Delta(G)$ is a complete graph of order $k+1$;*

(2) *if $k \geq 4$ is even, then $\Delta(G)$ is either a complete graph of order $k+1$ or a k-regular graph of order $k+2$.*

Zuccari [19] showed that if the prime graph $\Delta(G)$ of a finite solvable group G is a noncomplete regular graph with $|\rho(G)| = n$, then $\Delta(G)$ is $(n-2)$-regular. In particular, this answered Tong-Viet's conjecture when G is a solvable group. Sayanjali, Akhlaghi and Khosravi [21] proved that $\Delta(G)$ is k-regular for some integer k if and only if $k = 0$ or $\overline{\Delta}(G)$ is a regular bipartite graph, and in particular, if $k \neq |\rho(G)| - 1$, then $|\rho(G)|$ is even. This also provides some evidence for Tong-Viet's conjecture. However, the conjecture has not been proved for finite nonsolvable groups.

In this paper, we only deal with finite nonsolvable groups. In fact, we obtain the following result.

Theorem 1.1. *Let G be a nonsolvable group. If the prime graph $\Delta(G)$ of G is 5-regular, then $\Delta(G)$ is isomorphic to K_6 or possibly the graph depicted in Figure 1* *(see Section 4). Moreover, if G is an almost simple group, then $\Delta(G)$ is isomorphic to K_6.*

Theorem 1.1 provides some support for Conjecture 1 when $k = 5$. The graph depicted in Figure 1 is possible because it does not occur as the prime graph $\Delta(G)$ of any almost simple group G, but we have no way to rule it out, nor have we been able to construct a nonsolvable group G whose prime graph is isomorphic to it. It is easy to obtain a finite nonsolvable group whose prime graph is isomorphic to K_6; for example, see [27, Sections 2–5].

Question 1.2. Decide whether or not the graph depicted in Figure 1 can occur as the prime graph of a finite nonsolvable group. If it is possible, what can we say about those finite nonsolvable groups?

All further unexplained notation and terminology are standard, and we refer the readers to [3,9], if necessary.

2 Preliminaries

In this section, for the sake of convenience, we will prove a lemma and state some known results from the literature which will be used in the remainder of this article.

Lemma 2.1 (see [23, Corollary 2.5]). *Let G be a group with prime graph $\Delta(G)$. Suppose that the maximal degree of $\Delta(G)$ is $d \geq 3$ and $\Delta(G)$ is K_{d+1}-free. Then $|\rho(G)| \leq 3d$, and if G is solvable, then $|\rho(G)| \leq 2d$. In particular, if $\Delta(G)$ is connected k-regular for some $k \geq 3$ which is not K_{k+1}, then $|\rho(G)| \leq 3k$, and if G is solvable, then $|\rho(G)| \leq 2k$.*

Lemma 2.2 (see [3, Corollary 1.2]). *In any graph, the number of vertices of odd degree is even.*

Lemma 2.3 ([1, Theorem A]). *Let G be a finite group. Then the following conclusions hold:*

(a) *if $\omega(\Delta(G)) \leq 5$, then $|\rho(G)| \leq 2\omega(\Delta(G)) + 1$;*

(b) *if $\omega(\Delta(G)) \geq 5$, then $|\rho(G)| \leq 3\omega(\Delta(G)) - 4$.*

Lemma 2.4 (see [9, Corollary 11.29]). *Let N be a normal subgroup of a group G and $\chi \in \mathrm{Irr}(G)$. If θ is an irreducible constituent of χ_N, then $\chi(1)/\theta(1) \mid |G : N|$.*

Lemma 2.5 (see [22, Lemma 4.2]). *Let N be a normal subgroup of a group G such that $G/N \cong S$, where S is a nonabelian simple group. If $\theta \in \mathrm{Irr}(N)$, then either $\chi(1)/\theta(1)$ is divisible by two distinct primes in $\pi(G/N)$ for some $\chi \in \mathrm{Irr}(\theta^G)$ or θ extends to $\theta_0 \in \mathrm{Irr}(G)$ and $G/N \cong \mathrm{A}_5$ or $\mathrm{L}_2(8)$.*

Lemma 2.6 (Gallagher's Theorem; see [9, Corollary 6.17]). *Let N be a normal subgroup of a group G. If $\theta \in \mathrm{Irr}(N)$ can extend to $\theta_0 \in \mathrm{Irr}(G)$, then the characters $\beta\theta_0$ for $\beta \in \mathrm{Irr}(G/N)$ are irreducible, distinct for distinct β and are all of the irreducible constituents of θ^G.*

Remark 2.7. Under the hypotheses of Lemma 2.5, from Lemmas 2.5 and 2.6, a useful consequence is that if $\theta \in \mathrm{Irr}(N)$ with $p \mid \theta(1)$ and $p \nmid |G/N|$, then either p is adjacent to at least two distinct primes in $\pi(G/N)$, or each prime in $\pi(\theta(1))$ is adjacent to each prime in $\pi(G/N)$ and $G/N \cong \mathrm{A}_5$ or $\mathrm{L}_2(8)$.

Lemma 2.8. *Let G be a group and N a normal subgroup of G. Then*

$$\rho(G) \setminus \pi(G/N) = \rho(N) \setminus \pi(G/N).$$

Proof. Let $p \in \rho(G) \setminus \pi(G/N)$. Then there exists a nonlinear irreducible character χ of G such that $p \mid \chi(1)$. Let θ be an irreducible constituent of χ_N. By Lemma 2.4, we have that $\chi(1)/\theta(1) \mid |G/N|$. Since $(p, |G/N|) = 1$, we deduce that $p \mid \theta(1)$, so $\rho(G) \setminus \pi(G/N) \subseteq \rho(N) \setminus \pi(G/N)$. Clearly,

$$\rho(N) \setminus \pi(G/N) \subseteq \rho(G) \setminus \pi(G/N).$$

The proof is complete. □

Lemma 2.9 (see [2, Lemma 2.3]). *Let S be a nonabelian simple group and G an almost simple group with $S \trianglelefteq G \leq \mathrm{Aut}(S)$. Suppose that p and q are primes such that q lies in $\rho(S)$, whereas p lies in $\rho(G) \setminus \rho(S)$. If p and q are not adjacent in $\Delta(G)$, then S is a simple group of Lie type in characteristic q.*

Lemma 2.10 (see [2, Lemma 2.4]). *Let S be a nonabelian simple group and G an almost simple group with $S \trianglelefteq G \leq \mathrm{Aut}(S)$. If p and q are distinct prime divisors of $|G|$ both coprime to $|S|$, then p and q are adjacent in $\Delta(G)$.*

Lemma 2.11 (see [2, Lemma 2.5]). *Let S be a nonabelian simple group. If the prime graph $\Delta(S)$ is K_5-free, then S is isomorphic to one of the following groups:*

(1) M_{11}, M_{12}, J_1 *and* J_2*;*

(2) A_n *with* $5 \leq n \leq 10$*;*

(3) $\mathrm{L}_2(q)$ *with* $q = p^f$ *and* $|\pi(q \pm 1)| \leq 4$*, where* p *is a prime;*

(4) $\mathrm{L}_3(q)$ *with* $|\pi(S)| \leq 5$*;*

(5) $\mathrm{U}_3(q)$ *with* $|\pi(S)| \leq 5$*;*

(6) $\mathrm{Sz}(q)$ *with* $q = 2^{2n+1}$ *and* $|\pi(S)| \leq 5$*;*

(7) *the remaining simple groups of Lie type with order divisible by at most four primes.*

Lemma 2.12 (Pálfy's Condition; see [20, Theorem]). *Let G be a solvable group and Π a subset of $\rho(G)$ with $|\Pi| = 3$. Then there exists an irreducible character of G with degree divisible by at least two primes from Π.*

3 Almost simple groups

Itô–Michler's Theorem ([9, Theorem 6.15] and [16, Theorem 5.4]), a fundamental theorem on character degrees, says that the prime p does not divide every nonlinear character degree of a group G if and only if G has normal abelian Sylow p-subgroups. In particular, $\rho(G) = \pi(G)$ for any almost simple group G since G has no nontrivial normal abelian Sylow subgroups. This fact will be used frequently without any further reference.

In this section, we complete the proof of the second assertion in Theorem 1.1 independently. We will break the proof into several lemmas.

Theorem 3.1. *Let G be an almost simple group. If the prime graph $\Delta(G)$ of G is 5-regular, then $\Delta(G)$ is isomorphic to K_6.*

Proof. If $|\rho(G)| = 6$, then we are done. Now suppose that $|\rho(G)| > 6$. By [23, Lemma 2.6], it is obvious that $\Delta(G)$ is K_6-free. Let S be the socle of G. Since $S \trianglelefteq G$, we have that $\Delta(S)$ is a subgraph of $\Delta(G)$, so $\Delta(S)$ is K_6-free. Then S is isomorphic to one of the groups listed in Lemma 3.2. It follows from Lemmas 3.3–3.9 that $\Delta(G)$ cannot be 5-regular, a contradiction. The proof is complete. □

We first prove the following lemma, which may be useful in other purposes as well. In the proof of Lemma 3.2, we need to consider the number of distinct primes of the orders of finite nonabelian simple groups. Finite nonabelian simple groups with orders divisible by at most four primes are classified in [8]. Finite nonabelian simple groups with orders divisible by five or six primes are classified in [10].

Lemma 3.2. *Let S be a nonabelian simple group. If the prime graph $\Delta(S)$ is K_6-free, then S is isomorphic to one of the following groups:*

(1) J_1, J_2, J_3, M_{11}, M_{12}, M_{22}, M_{23}, HS, He *and* McL*;*

(2) A_n *for* $n \in \{5, 6, 7, 8, 9, 10, 11, 12\}$*;*

(3) $\mathrm{L}_4(q)$ *for* $q \in \{3,4,5,7\}$, $\mathrm{L}_5(2)$, $\mathrm{L}_5(3)$, $\mathrm{L}_6(2)$, $\mathrm{O}_7(3)$, $\mathrm{S}_4(q)$ *for* $q \in \{3,4,5,7,9\}$, $\mathrm{S}_6(2)$, $\mathrm{S}_6(3)$, $\mathrm{S}_8(2) \cong \mathrm{O}_9(2)$, $\mathrm{U}_4(q)$ *for* $q \in \{3,4,5,7,9\}$, $\mathrm{U}_5(2)$, $\mathrm{U}_5(3)$, $\mathrm{U}_6(2)$, $\mathrm{O}_8^+(2)$, $\mathrm{O}_8^-(2)$, $\mathrm{O}_8^+(3)$, ${}^2\mathrm{F}_4(2)'$, $\mathrm{G}_2(q)$ *for* $q \in \{3,4,5,7,9\}$, ${}^3\mathrm{D}_4(2)$, ${}^3\mathrm{D}_4(3)$*;*

(4) $\mathrm{Sz}(q)$*, where* $q = 2^{2n+1} \geq 8$ *and* $|\pi((q-1)(q^2+1))| \leq 5$*;*

(5) $\mathrm{O}_5(q)$*, where* $q = p^f$*,* p *a prime and* $|\pi(q^4-1)| = 4$*;*

(6) $\mathrm{R}(q)$*, where* $q = 3^{2n+1}$*,* $|\pi(q^2-1)| = 3$ *and* $|\pi(q^2-q+1)| = 1$*;*

(7) $\mathrm{L}_2(q)$*, where* $q = p^f$*,* p *a prime and* $|\pi(q \pm 1)| \leq 5$*;*

(8) $\mathrm{L}_3(q)$*, where* $q = p^f$*,* p *a prime and* $|\pi(q^2-1)(q^3-1)| \leq 4$ *or* $|\pi(q^2-1)(q^3-1)| = 5$*; in the latter case, if* q *is odd, then* $q-1$ *is divisible by a prime other than* 2 *or* 3*;*

(9) $\mathrm{U}_3(q)$*, where* $q = p^f$*,* p *a prime and* $|\pi(q^2-1)(q^3+1)| \leq 4$ *or* $|\pi(q^2-1)(q^3+1)| = 5$*; in the latter case,* $q+1$ *is divisible by a prime other than* 2 *or* 3*.*

Proof. If $3 \leq |\pi(S)| \leq 5$, then $\Delta(S)$ is K_6-free since $\rho(S) = \pi(S)$.

- If $|\pi(S)| = 3$, then S is isomorphic to one of the following groups: A_5, A_6, $\mathrm{S}_4(3) \cong \mathrm{U}_4(2)$, $\mathrm{L}_2(7)$, $\mathrm{L}_2(8)$, $\mathrm{U}_3(3)$, $\mathrm{L}_3(3)$ and $\mathrm{L}_2(17)$ by [8, Table 1]. These groups appear somewhere in the conclusion of the lemma.
- If $|\pi(S)| = 4$, then S is isomorphic to one of the following groups: J_2, A_n for $n \in \{7,8,9,10\}$, $\mathrm{L}_3(q)$ for $q \in \{4,5,7,8,17\}$, $\mathrm{U}_3(q)$ for $q \in \{4,5,7,8,9\}$, $\mathrm{U}_4(3)$, $\mathrm{S}_4(q)$ for $q \in \{4,5,7,9\}$, $\mathrm{S}_6(2) \cong \mathrm{O}_7(2)$, $\mathrm{O}_8^+(2)$, M_{11}, M_{12}, $\mathrm{U}_5(2)$, $\mathrm{L}_4(3)$, ${}^2\mathrm{F}_4(2)'$, $\mathrm{G}_2(3)$, ${}^3\mathrm{D}_4(2)$, $\mathrm{Sz}(8)$, $\mathrm{Sz}(32)$ and $\mathrm{L}_2(q)$ for $|\pi(q^2-1)| = 3$ by [8, Theorem I]. These groups appear somewhere in the conclusion of the lemma.
- If $|\pi(S)| = 5$, then S is isomorphic to one of the following groups: A_{11}, A_{12}, M_{22}, J_3, HS, He, McL, $\mathrm{L}_4(q)$ for $q \in \{4,5,7\}$, $\mathrm{L}_5(2)$, $\mathrm{L}_5(3)$, $\mathrm{L}_6(2)$, $\mathrm{O}_7(3)$, $\mathrm{S}_6(3)$, $\mathrm{S}_8(2) \cong \mathrm{O}_9(2)$, $\mathrm{U}_4(q)$ for $q \in \{4,5,7,9\}$, $\mathrm{U}_5(3)$, $\mathrm{U}_6(2)$, $\mathrm{O}_8^+(3)$, $\mathrm{O}_8^-(2)$, ${}^3\mathrm{D}_4(3)$, $\mathrm{G}_2(q)$ for $q \in \{4,5,7,9\}$, $\mathrm{L}_2(q)$ for $|\pi(q^2-1)| = 4$, $\mathrm{L}_3(q)$ for $|\pi(q^2-1)(q^3-1)| = 4$, $\mathrm{U}_3(q)$ for $|\pi(q^2-1)(q^3+1)| = 4$, $\mathrm{O}_5(q)$ for $|\pi(q^4-1)| = 4$, $\mathrm{Sz}(2^{2n+1})$ for $|\pi((2^{2n+1}-1)(2^{4n+2}+1))| = 4$ and $\mathrm{R}(3^{2n+1})$, where $|\pi(3^{4n+2}-1)| = 3$ and $|\pi(3^{4n+2}-3^{2n+1}+1)| = 1$ by [10, Theorem A]. These groups appear somewhere in the conclusion of the lemma.
- If $|\pi(S)| = 6$, by [10, Theorem B] and [27, Sections 2–5], S is isomorphic to one of the following groups: M_{23}, J_1, $\mathrm{L}_2(q)$ for $|\pi(q^2-1)| = 5$,

$L_3(q)$ for $|\pi(q^2-1)(q^3-1)| = 5$ and if q is odd, then $q-1$ is divisible by a prime other than 2 or 3, $U_3(q)$ for $|\pi(q^2-1)(q^3+1)| = 5$ and $q+1$ is divisible by a prime other than 2 or 3, and $Sz(2^{2n+1})$ for $|\pi((2^{2n+1}-1)(2^{4n+2}+1))| = 5$. These groups appear somewhere in the conclusion of the lemma.

- If $|\pi(S)| \geq 7$, by [27, Sections 2–5], we have that S is isomorphic to $L_2(q)$ for $|\pi(q \pm 1)| \leq 5$. These groups appear somewhere in the conclusion of the lemma.

The proof is complete. □

We will assume in Lemmas 3.3–3.9 that S is a nonabelian simple group and G an almost simple group with $S \trianglelefteq G \leq \mathrm{Aut}(S)$.

Lemma 3.3. *If S is isomorphic to one of the groups listed in Lemma 3.2 (1)–(3), then $\Delta(G)$ cannot be 5-regular.*

Proof. In cases (1)–(3) of Lemma 3.2, the order of the outer automorphism group of S is one of the integers in $\{1, 2, 3, 4, 6, 8, 24\}$ by Atlas [4] or [29]. We have $\rho(G) = \pi(G) = \pi(S)$ since $6 \mid |S|$. We know that $|\pi(S)| \leq 5$ except J_1 and M_{23}. By Atlas [4], $|\mathrm{Out}(J_1)| = |\mathrm{Out}(M_{23})| = 1$, $|\pi(J_1)| = |\pi(M_{23})| = 6$ and both of their prime graphs are not 5-regular. The proof is complete. □

Lemma 3.4. *If S is isomorphic to one of the groups listed in Lemma 3.2 (4), then $\Delta(G)$ cannot be 5-regular.*

Proof. The only outer automorphism of $Sz(q)$ is the field automorphism. Let $r = |G : S|$. Then $\rho(G) = \{2\} \cup \pi(r(q-1)(q^2+1))$. By [17, Theorem 6.5], the subgraph corresponding to $\pi(r(q-1)(q^2+1))$ is complete, and 2 is adjacent to precisely those primes dividing $q-1$. Therefore, $\Delta(G)$ cannot be 5-regular. The proof is complete. □

Lemma 3.5. *If S is isomorphic to one of the groups listed in Lemma 3.2 (5), then either $\Delta(G)$ is isomorphic to K_6 or $\Delta(G)$ cannot be 5-regular.*

Proof. By [27, Theorem 5.7], the subgraph corresponding to $\pi(p(q^4-1))$ is complete. Let $r = |G : S|$. By Lemmas 2.9 and 2.10, the subgraph corresponding to $\pi(r(q^4-1))$ is complete. If $|\pi(pr(q^4-1))| = 5$, then $\Delta(G)$ cannot be 5-regular. If $|\pi(pr(q^4-1))| = 6$, then either $\Delta(G)$ is isomorphic to K_6 or $K_6 - e$, where e is an edge of K_6. If $|\pi(pr(q^4-1))| \geq 7$, then $\Delta(G)$ cannot be 5-regular. The proof is complete. □

Lemma 3.6. *If S is isomorphic to one of the groups listed in Lemma 3.2 (6), then either $\Delta(G)$ is isomorphic to K_6 or $\Delta(G)$ cannot be 5-regular.*

Proof. This is achieved by similar arguments to those in the proof of Lemma 3.5 since the subgraph corresponding to $\pi(3(q^3+1)(q-1))$ is complete by [27, Theorem 4.2]. □

Lemma 3.7. *If S is isomorphic to one of the groups listed in Lemma 3.2 (7), then either $\Delta(G)$ is isomorphic to K_6 or $\Delta(G)$ cannot be 5-regular.*

Proof. Suppose that $\Delta(G)$ is 5-regular and let $\Omega = \pi(G) \setminus \pi(S)$. We claim that $|\pi(G)| \leq 11$. If $\Omega \neq \emptyset$, then $|\pi((q-1)(q+1))| \leq 5$ and $|\pi(G) \setminus \pi(S)| \leq 5$ by Lemmas 2.9 and 2.10, and so $|\pi(G)| \leq 11$. If $\Omega = \emptyset$, clearly $|\pi(G)| \leq 11$. The claim holds. By Lemma 2.2, we obtain that $|\pi(G)| \in \{6, 8, 10\}$. If $|\pi(G)| = 6$, then we are done. From now on, we assume that $|\pi(G)| \in \{8, 10\}$. If $\Omega \neq \emptyset$, then there is no way to guarantee that $\Delta(G)$ is 5-regular by Lemmas 2.9 and 2.10, a contradiction. So we may assume that $\pi(G) = \pi(S)$.

Case 1: $S \cong \mathrm{L}_2(2^f)$ and $|\pi(2^f \pm 1)| \leq 5$. Let $|G : S| = r = 2^a m$, where $a \geq 0$ and m is odd. If $m \neq 1$, then by [28, Theorem A], we have that $m(2^f \pm 1)$ divides some degree in $\mathrm{cd}(G)$, and so every prime in $\pi(m)$ has degree greater than or equal to 6, a contradiction. From now on, we assume that $m = 1$. If $a = 0$, then $G = S$, and so $\Delta(G)$ cannot be 5-regular by [27, Theorem 5.2], a contradiction. Hence we have that $|G : S| = 2^a$ with $a \geq 1$. If $f \equiv 2 \pmod 4$ and $G = \mathrm{Aut}(S)$ are not both true, then by [28, Theorem A], we know that $2(2^f \pm 1)$ divides some degree in $\mathrm{cd}(G)$, and so 2 is adjacent to all primes in $\pi(G) \setminus \{2\}$, a contradiction. So we may suppose that $f \equiv 2 \pmod 4$ and $G = \mathrm{Aut}(S)$. If $a \neq 1$, then by [28, Theorem A], we deduce that $2(2^f \pm 1)$ divides some degree in $\mathrm{cd}(G)$, and so 2 is adjacent to all primes in $\pi(G) \setminus \{2\}$, a contradiction. If $a = 1$, then $S \cong \mathrm{L}_2(4) \cong \mathrm{A}_5$ and $G \cong \mathrm{S}_5$, and thereby $|\pi(G)| \notin \{8, 10\}$, a contradiction.

Case 2: $S \cong \mathrm{L}_2(q)$, where $q = p^f$, p an odd prime and $|\pi(q \pm 1)| \leq 5$. In this case, by [27, Theorem 5.2], 2 is adjacent to all primes in $\pi((q-1)(q+1)) \setminus \{2\}$ and so $\deg(2) \geq 6$, a contradiction.

The proof is complete. □

Lemma 3.8. *If S is isomorphic to one of the groups listed in Lemma 3.2 (8), then either $\Delta(G)$ is isomorphic to K_6 or $\Delta(G)$ cannot be 5-regular.*

Proof. Suppose that $\Delta(G)$ is 5-regular. By Lemmas 2.9 and 2.10, we deduce that $|\pi(G)| \leq 7$. It follows from Lemma 2.2 that $|\pi(G)| = 6$, as desired. The proof is complete. □

Lemma 3.9. *If S is isomorphic to one of the groups listed in Lemma 3.2 (9), then either $\Delta(G)$ is isomorphic to K_6 or $\Delta(G)$ cannot be 5-regular.*

Proof. This is achieved by similar arguments to those in the proof of Lemma 3.8. □

4 Proof of Theorem 1.1

In this section, we complete the proof of the first assertion in Theorem 1.1. We will break the proof into several lemmas. We mainly use Clifford Theory and Gallagher's Extension Theorem; one may refer to [9, Chapter 6]. We will prove the graph depicted in Figure 2 cannot occur as the prime graph of a finite nonsolvable group.

Lemma 4.1. *If the prime graph $\Delta(G)$ of a group G is a 5-regular graph with $|\rho(G)| > 6$, then $\Delta(G)$ is isomorphic to one of the graphs depicted in Figures 1 and 2.*

Proof. By [23, Lemma 2.6], we have that $\Delta(G)$ is connected 5-regular, and so $\Delta(G)$ is K_6-free. By Lemmas 2.1 and 2.2, we have that $|\rho(G)| \in \{8, 10, 12, 14\}$. Write $\rho(G) = \{p_i\}_{i=1}^{n}$, where $n = |\rho(G)|$.

Case 1: $\Delta(G)$ is a 5-regular graph of order 8. By Lemma 2.3, we obtain that $\Delta(G)$ must contain a complete subgraph K_4 with vertex set, say $\{p_1, p_2, p_3, p_4\}$. Each prime in $\{p_i\}_{i=5}^{8}$ is adjacent to at least two distinct primes in $\{p_i\}_{i=1}^{4}$ since $\Delta(G)$ is a 5-regular graph.

Subcase 1.1: Suppose that some prime p_j in $\{p_i\}_{i=5}^{8}$, say p_5, is adjacent to all primes in $\{p_1, p_2, p_3, p_4\}$. Now $\deg(p_i) = 4$ for $i \in \{1, 2, 3, 4\}$. There is no way to guarantee that every prime in $\{p_6, p_7, p_8\}$ is adjacent to at least two distinct primes in $\{p_i\}_{i=1}^{4}$. Hence this subcase cannot happen.

Subcase 1.2: Suppose that some p_j in $\{p_i\}_{i=5}^{8}$, say p_5, is adjacent to three distinct primes in $\{p_i\}_{i=1}^{4}$, say p_1, p_2 and p_3. Now $\deg(p_i) = 4$ for $i \in \{1, 2, 3\}$. Again, there is no way to guarantee that every prime in $\{p_6, p_7, p_8\}$ is adjacent to at least two distinct vertices in $\{p_i\}_{i=1}^{4}$. Hence this subcase cannot happen.

Subcase 1.3: Suppose that some p_j in $\{p_i\}_{i=5}^{8}$, say p_5, is adjacent to two distinct vertices in $\{p_i\}_{i=1}^{4}$, say p_1 and p_2. In fact, in this subcase, each prime in $\{p_i\}_{i=5}^{8}$ is adjacent to exactly two distinct vertices in $\{p_i\}_{i=1}^{4}$, and $\{p_i\}_{i=5}^{8}$ induces a complete subgraph K_4. Only two graphs (Figures 1 and 2) satisfy the conditions, up to isomorphism.

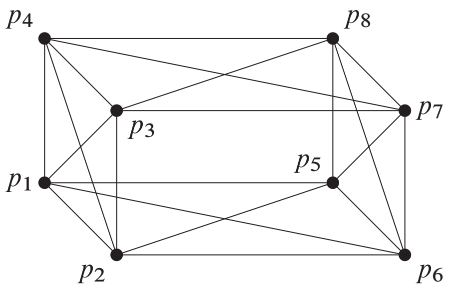

Figure 1. 5-regular graph of order 8 with 4 K_4's

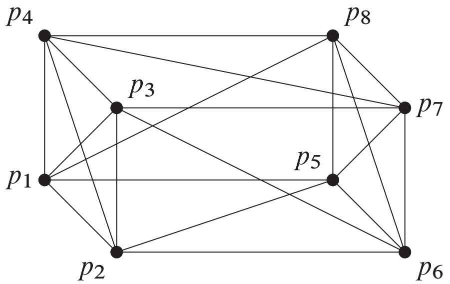

Figure 2. 5-regular graph of order 8 with 2 K_4's

Case 2: $\Delta(G)$ is a 5-regular graph of order 10. By Lemma 2.3, we have that $\Delta(G)$ must contain a complete subgraph K_5 with vertex set, say $\{p_1, p_2, p_3, p_4, p_5\}$. Each prime in $\{p_i\}_{i=6}^{10}$ is adjacent to at least one prime in $\{p_i\}_{i=1}^{5}$ since $\Delta(G)$ is a 5-regular graph. Only one graph (Figure 3) satisfies the conditions, up to isomorphism. This is in contradiction to [5, Theorem 3.1].

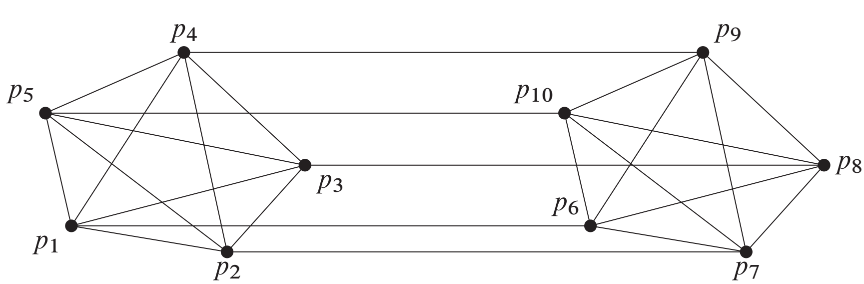

Figure 3. 5-regular graph of order 10 with 2 K_5's

Case 3: $\Delta(G)$ is a 5-regular graph of order 12 or 14. By Lemma 2.3, we know that $\Delta(G)$ must contain a complete subgraph K_6, a contradiction.

The proof is now complete. □

Theorem 4.2. *Let G be a nonsolvable group. If the prime graph $\Delta(G)$ is isomorphic to the graph depicted in Figure 2, then such a G does not exist.*

Proof. This follows directly from Hypothesis 4.5, Remark 4.6, Corollary 4.11, Lemmas 4.12, 4.14 and 4.16. □

Lemma 4.3. *Let G be a nonsolvable group. If the prime graph $\Delta(G)$ is 5-regular with $|\rho(G)| > 6$, then every nonsolvable chief factor of G is simple.*

Proof. By hypotheses and Lemma 4.1, we obtain that $\Delta(G)$ is isomorphic to one of the graphs depicted in Figures 1 and 2, and so $\Delta(G)$ is K_5-free. If M/N is a nonsolvable chief factor of G, then $M/N \cong S^k$ for some nonabelian simple group S and some integer $k \geq 1$. Let $C/N = C_{G/N}(M/N)$. Then $C \trianglelefteq G$ and G/C has the unique minimal normal subgroup MC/C which is isomorphic to M/N.

We claim that $k = 1$. Assume by contradiction that $k \geq 2$. Since G/C has no nontrivial normal abelian Sylow subgroups, by Itô–Michler's Theorem, we have that $\rho(G/C) = \pi(G/C)$ and thus $|\rho(G/C)| = |\pi(G/C)| \geq 3$. By [12, Main Theorem], we know that $\Delta(G/C)$ is complete, and thus

$$|\rho(G/C)| = |\pi(G/C)| \leq 4$$

since $\Delta(G/C)$ is K_5-free. By Lemma 2.8, we have that

$$\Omega := \rho(G) \setminus \rho(G/C) \subseteq \rho(C).$$

Thus there exists $\theta \in \mathrm{Irr}(C)$ such that $r \mid \theta(1)$, where $r \in \Omega$. Let L be a normal subgroup of MC such that $L/C \cong S$. Write

$$\rho(G/C) = \{p_i\}_{i=1}^{n}, \quad \text{where } n = |\rho(G/C)|.$$

We prove the lemma case by case.

Case 1: $n = 3$. Then $|\Omega| = 5$. Let $\Omega = \{x_i\}_{i=1}^{5}$ and $x_i \in \Omega$ with $x_i \mid \theta_i(1)$, where $\theta_i \in \mathrm{Irr}(C)$. By Lemma 2.5, either $\chi(1)/\theta_i(1)$ is divisible by two distinct primes in $\pi(L/C)$ for some $\chi \in \mathrm{Irr}(\theta_i^L)$ or θ_i extends to $(\theta_i)_0 \in \mathrm{Irr}(L)$. If the former case holds, then x_i is adjacent to at least two distinct primes in $\pi(L/C)$. If the latter case holds, then x_i is adjacent to all primes in $\pi(L/C)$. Therefore, x_i is adjacent to at least two distinct primes in $\pi(L/C)$. Hence $\deg(p_i) \geq 6$ for some $i \in \{1, 2, 3\}$, a contradiction.

Case 2: $n = 4$. Since $\Delta(G)$ is isomorphic to one of the graphs depicted in Figures 1 and 2, we have that all primes in Ω must induce a complete subgraph K_4. Assume without loss of generality that $\Omega = \{p_i\}_{i=5}^{8}$ (the letters here correspond to the letters in Figures 1 and 2). In both cases, there exists $\psi \in \mathrm{Irr}(G)$ such that $p_6 p_7 \mid \psi(1)$. Let φ be an irreducible constituent of ψ_C. By Lemma 2.4, we obtain that $\psi(1)/\varphi(1) \mid |G : C|$. Since $(p_6 p_7, |G : C|) = 1$, we have that $p_6 p_7 \mid \varphi(1)$. By Lemma 2.5, either $\chi(1)/\varphi(1)$ is divisible by two distinct primes in $\pi(L/C)$ for some $\chi \in \mathrm{Irr}(\varphi^L)$, or φ extends to $\varphi_0 \in \mathrm{Irr}(L)$. If the former case holds, then $\Delta(L)$ would contain a complete subgraph K_4 with vertex set that contains p_6, p_7 and two of primes in $\{p_i\}_{i=1}^{4}$, and so does $\Delta(G)$, a contradiction. If the latter case holds, then p_6 is adjacent to all primes in $\pi(L/C) \subseteq \{p_i\}_{i=1}^{4}$, which is impossible since $|\pi(L/C)| \geq 3$.

The claim holds, and so the proof is complete. □

Lemma 4.4. *Let G be a group and $\Delta(G)$ isomorphic to the graph depicted in Figure 2. If N is the solvable radical of G, and M/N is a chief factor of G, then G/N is almost simple with socle M/N.*

Proof. Since N is the solvable radical of G, we deduce that M/N is nonsolvable. It follows from Lemma 4.3 that $M/N \cong S$, where S is a nonabelian simple group. Let $C/N = C_{G/N}(M/N)$. Then G/C is almost simple with socle $MC/C \cong M/N$.

It suffices to show that $C = N$. Assume by contradiction that $N \lneqq C$. Let L/N be a minimal normal subgroup of G/N such that $L \leq C$. By Lemma 4.3 again, we obtain that $L/N \cong T$, where T is a nonabelian simple group. Note that $ML/N = M/N \times L/N$. We have that every prime in $\Omega := \rho(M/N) \cap \rho(L/N)$ is adjacent to all primes in $\Lambda := \rho(M/N) \cup \rho(L/N)$. In particular, 2 is adjacent to all primes in Λ. Therefore, $|\Lambda| \leq 6$. We claim that $|\rho(M/N)| \geq 4$. Assume by contradiction that $|\rho(M/N)| = 3$. By [8, Table 1] and Atlas [4], we deduce that $\rho(G/C) = \rho(MC/C) = \rho(M/N)$. It follows from Lemma 2.8 that $|\rho(C)| \geq 5$. By [18, Main Theorem], there exist at least two edges in $\rho(C)$. By Lemma 2.5, we deduce that $\Delta(MC)$ has a subgraph isomorphic to one of the graphs depicted in Figure 4. This is a contradiction since $\Delta(MC)$ is a subgraph of $\Delta(G)$. Hence the claim holds, and so $4 \leq |\Lambda| \leq 6$. We prove the lemma one by one.

Case 1: $|\Lambda| = 6$. From the structure of the graph depicted in Figure 2, any six vertices chosen in the graph contain at most one complete vertex. This forces $|\Omega| = 1$. Since $|\rho(M/N)| \geq 4$ and $|\rho(L/N)| \geq 3$, we have that $|\rho(M/N)| = 4$ and $|\rho(L/N)| = 3$. By [8, Theorem I], we obtain that M/N is isomorphic to Sz(8) or Sz(32). By [27, Theorem 4.1], we deduce that Λ has at least four complete vertices, a contradiction.

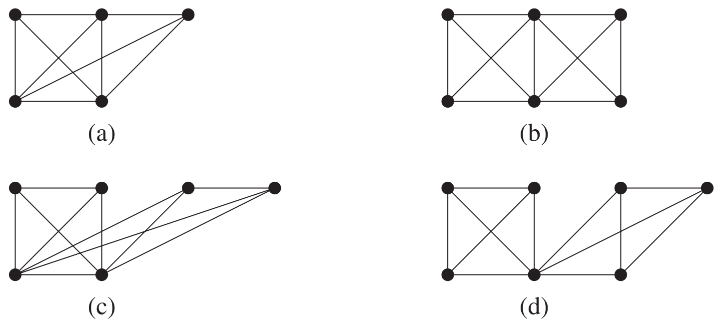

Figure 4. Some impossible subgraphs of $\Delta(MC)$

Case 2: $|\Lambda| = 5$. From the structure of the graph depicted in Figure 2, any five vertices chosen in the graph contain at most two complete vertices. This forces $|\Omega| \leq 2$. Since $|\rho(M/N)| \geq 4$ and $|\rho(L/N)| \geq 3$, we obtain that

$$|\Omega| = 2, \quad |\rho(M/N)| = 4 \quad \text{and} \quad |\rho(L/N)| = 3.$$

Hence Λ has at least three complete vertices, a contradiction.

Case 3: $|\Lambda| = 4$. Since we have that $|\rho(M/N)| \geq 4$ and $|\rho(L/N)| \geq 3$, it follows that $3 \leq |\Omega| \leq 4$. If $|\Omega| = 3$, then $|\rho(M/N)| = 4$. Notice that all primes in $\rho(M/N)$ induce a complete subgraph K_4 since $ML/N = M/N \times L/N$. If $MC/C \cong M/N$ is isomorphic to one of the groups listed in [8, Table 2] except Sz(8), then by Atlas [4], we have that $\rho(G/C) = \rho(M/N)$. It follows from Lemma 2.8 that $\rho(G) \setminus \rho(G/C) \subseteq \rho(C)$. By [18, Main Theorem], there exists at least one edge in $\rho(G) \setminus \rho(G/C)$. Let $\theta \in \mathrm{Irr}(C)$ such that $r_1 r_2 \mid \theta(1)$, where r_1 and r_2 belong to $\rho(G) \setminus \rho(G/C)$. By Lemma 2.5, we have that r_1, r_2 and two of the primes in $\rho(MC/C)$ induce a complete subgraph K_4. This implies that $\Delta(MC)$ has two K_4's sharing a common edge, a contradiction since $\Delta(MC)$ is a subgraph of $\Delta(G)$. If $MC/C \cong M/N$ is isomorphic to Sz(8), then by Atlas [4], we have that $|\rho(G/C)| = 4$ or $|\rho(G/C)| = 5$. The former case cannot occur using the same reasoning as above. So we may assume that $|\rho(G/C)| = 5$. Let

$$r \in \rho(G/C) \setminus \rho(MC/C).$$

By Lemma 2.9, we deduce that r is adjacent to at least three primes in $\rho(MC/C)$. This means that $\Delta(G/C)$ has a subgraph isomorphic to $K_5 - e$, where e is an edge of K_5. This is a contradiction since $\Delta(G/C)$ is a subgraph of $\Delta(G)$. Hence, by [8, Theorem I], we may assume that M/N is isomorphic to $\mathrm{L}_2(q)$ for some suitable prime power q. By [8, Theorem 3.2, Lemmas 3.4 and 3.5] and Atlas [4], we have that $|\rho(G/C)| = 4$ or $|\rho(G/C)| = 5$. By the same reasoning as before, we obtain a contradiction. If $|\Omega| = 4$, then $\rho(M/N) = \rho(L/N)$ and $|\rho(M/N)| = 4$. We also get a contradiction by the same reasoning as in $|\Omega| = 3$.

Therefore, $C = N$ and the proof is complete. □

It follows from Lemma 4.4 that we may assume the following hypothesis.

Hypothesis 4.5. Let G be a nonsolvable group such that $\Delta(G)$ is isomorphic to the graph depicted in Figure 2. Suppose that N is the solvable radical of G such that G/N is almost simple with socle $M/N \cong S$, a nonabelian simple group. Write $\Omega := \rho(G) \setminus \pi(G/N)$.

Remark 4.6. Under Hypothesis 4.5, since $\Delta(G)$ is K_5-free, we get that $\Delta(G/N)$ is K_5-free and so is $\Delta(M/N)$. Hence S is isomorphic to one of the groups listed in Lemma 2.11. By Lemma 2.8, we have that $\Omega = \rho(N) \setminus \pi(G/N)$.

Lemma 4.7. *Under Hypothesis 4.5, we have that $|\Omega| \leq 3$ and the subgraph of $\Delta(G)$ on Ω has at most one edge.*

Proof. First, we claim that any two edges in Ω have no common vertex in $\Delta(G)$. Suppose by contradiction that, without loss of generality, $\{p_1, p_2\}$ and $\{p_2, p_3\}$ are edges in $\Delta(G)$, where p_1, p_2 and p_3 belong to Ω. Then there exist irreducible characters χ_1 and χ_2 of G such that $p_1 p_2 \mid \chi_1(1)$ and $p_2 p_3 \mid \chi_2(1)$, respectively. Let θ_1 and θ_2 be irreducible constituents of $(\chi_1)_N$ and $(\chi_2)_N$, respectively. By Lemma 2.4, we have that $p_1 p_2 \mid \theta_1(1)$ and $p_2 p_3 \mid \theta_2(1)$. By Lemma 2.5, we obtain that $\Delta(M)$ has a subgraph isomorphic to two K_4's sharing a common vertex p_2 and so does $\Delta(G)$, a contradiction. Hence the claim holds.

Assume by contradiction that $|\Omega| \geq 4$; then $|\Omega| = 4$ or 5 since $|\pi(G/N)| \geq 3$. If $|\Omega| = 5$, then by Lemma 2.12, there exist at least three edges in Ω, and so at least two edges have a common vertex, a contradiction. If $|\Omega| = 4$, then by Lemma 2.12 and the claim as above, there exist exactly two nonincident edges in Ω and also in $\rho(G)$. By Lemma 2.5, there is no way to guarantee that $\Delta(G)$ is isomorphic to the graph depicted in Figure 2, a contradiction. The proof is complete. □

Corollary 4.8. *Under Hypothesis 4.5, if $|\pi(S)| = 3$, then such a G does not exist.*

Proof. This follows directly from Lemma 4.7 and Atlas [4]. □

Lemma 4.9. *Under Hypothesis 4.5, if $|\pi(S)| = 4$ and the subgraph $\Delta(S)$ of $\Delta(G)$ is complete, then such a G does not exist.*

Proof. By [8, Theorems I and 3.2, Lemmas 3.4 and 3.5] and Atlas [4], we have that $|\rho(G/N)| = 4$ or $|\rho(G/N)| = 5$. By Lemma 4.7, the former case cannot occur. Hence $|\Omega| = 3$, and by Lemma 2.12, there exists one edge in Ω. By Lemma 2.5, we have that two of the primes in Ω and two of the primes in $\pi(S)$ induce a complete subgraph K_4 in $\Delta(M)$, and so also in $\Delta(G)$. Since all primes in $\pi(S)$ induce a complete subgraph K_4, we obtain that $\Delta(G)$ has a subgraph isomorphic to two K_4's sharing two common vertices, a contradiction. The proof is complete. □

Lemma 4.10. *Under Hypothesis 4.5, if $S \cong \mathrm{J}_1$, then such a G does not exist.*

Proof. By Atlas [4], we have that $G/N \cong \mathrm{J}_1$, and so $\Omega = \{p_1, p_2\}$. Let $\chi \in \mathrm{Irr}(G)$ such that $p_1 \mid \chi(1)$. Assume ψ is an irreducible constituent of χ_N and $I = I_G(\psi)$; then we have that $p_1 \mid \psi(1)$ by Lemma 2.4. It follows from Clifford Theory that there exists $\theta \in \mathrm{Irr}(\psi^I)$ such that $\chi = \theta^G$. Thus $\chi(1) = |G : I|e\psi(1)$, where $e = [\psi, \theta_N]$. By Atlas [4], we know that the Schur multiplier of J_1 is trivial. If $I = G$, then by [9, Theorem 11.7], we have that ψ extends to G. By Lemma 2.6, we obtain that p_1 is adjacent to all primes in $\pi(G/N)$, and so $\deg(p_1) \geq 6$, a contradiction. If $I \lneqq G$, then there exists a maximal subgroup K/N of G/N such that $I/N \leq K/N$. Hence $|G/N : K/N| = |G : K| \mid |G : I|$; this means that some index of maximal subgroup in J_1 divides $|G : I|$. Using Atlas [4], there are 7 possible values for $|G : K|$; they are

- $266 = 2 \cdot 7 \cdot 19$,
- $1045 = 5 \cdot 11 \cdot 19$,
- $1463 = 7 \cdot 11 \cdot 19$,
- $1540 = 2^2 \cdot 5 \cdot 7 \cdot 11$,
- $1596 = 2^2 \cdot 3 \cdot 7 \cdot 19$,
- $2926 = 2 \cdot 7 \cdot 11 \cdot 19$, and
- $4180 = 2^2 \cdot 5 \cdot 11 \cdot 19$.

The last four cases cannot occur; otherwise, $\Delta(G)$ has a subgraph isomorphic to K_5. For the first three cases, $\Delta(G)$ must have a subgraph isomorphic to K_4, which contains both p_1 and 19. By the same reasoning as above, $\Delta(G)$ has another subgraph isomorphic to K_4, which contains both p_2 and 19. Therefore, $\Delta(G)$ has a subgraph isomorphic to two K_4's sharing a common vertex, a contradiction. The proof is complete. □

Corollary 4.11. *Under Hypothesis 4.5, S cannot be isomorphic to one of the groups listed in Lemma 2.11* (1)*,* (2) *and* (7)*.*

Proof. This follows directly from Lemmas 4.7, 4.9 and 4.10, Atlas [4] and [27, Sections 4 and 5]. □

Lemma 4.12. *Under Hypothesis* 4.5*, if* S *is isomorphic to one of the groups listed in Lemma* 2.11 (4)–(6)*, then such a* G *does not exist.*

Proof. Suppose that $S \cong \mathrm{Sz}(q)$ with $q = 2^{2n+1}$ and $|\pi(S)| \leq 5$. Since the orders of the Suzuki simple groups are at least four distinct primes, we distinguish two cases.

Case 1: $|\pi(S)| = 4$. By [8, Table 2], we obtain that S is isomorphic to Sz(8) or Sz(32). Suppose that $S \cong \mathrm{Sz}(8)$; then $\pi(G/N) = \pi(S)$ or $\pi(G/N) = \pi(S) \cup \{3\}$ by Atlas [4]. If $\pi(G/N) = \pi(S)$, then $|\Omega| = 4$, a contradiction by Lemma 4.7. If $\pi(G/N) = \pi(S) \cup \{3\}$, then by Lemma 2.9, we obtain that 3 is adjacent to 5, 7 and 13 in $\Delta(G/N)$ and so also in $\Delta(G)$. By Atlas [4], we have that $\{5, 7, 13\}$ induces a triangle in $\Delta(S)$ and so also in $\Delta(G)$. Therefore, $\{3, 5, 7, 13\}$ induces a complete subgraph K_4 in $\Delta(G)$. As $|\Omega| = 3$, by Lemma 2.12, there exists an edge in Ω. By Lemma 4.3, we have that two of the primes in Ω and two of the primes in $\pi(S) = \{2, 5, 7, 13\}$ induce a complete subgraph K_4 in $\Delta(M)$ and so also in $\Delta(G)$. Hence $\Delta(G)$ has a subgraph isomorphic to two K_4's sharing a common vertex, a contradiction. Suppose that $S \cong \mathrm{Sz}(32)$; then $\pi(G/N) = \pi(S)$ by Atlas [4], and hence $|\Omega| = 4$, a contradiction by Lemma 4.7.

Case 2: $|\pi(S)| = 5$. By [27, Theorem 4.1], all primes in $\pi(S) \setminus \{2\}$ induce a complete subgraph of $\Delta(S)$ and so also of $\Delta(G)$. Suppose that $\pi(G/N) \setminus \pi(S) \neq \emptyset$ and let $r \in \pi(G/N) \setminus \pi(S)$. By Lemma 2.9, we have that r is adjacent to all primes in $\pi(S) \setminus \{2\}$ in $\Delta(G/N)$ and so also in $\Delta(G)$. Hence $\Delta(G)$ has a subgraph isomorphic to K_5, a contradiction. Therefore, $\pi(G/N) = \pi(S)$, and so $|\Omega| = 3$. By the same reasoning as in Case 1, we can get a contradiction.

Suppose that $S \cong \mathrm{L}_3(q)$ with $q = p^f$ and $|\pi(S)| \leq 5$. If $3 \leq |\pi(S)| \leq 4$, then by Atlas [4], we have that $\pi(S) = \pi(G/N)$, and so $|\Omega| \geq 4$, a contradiction by Lemma 4.7. If $|\pi(S)| = 5$, then by [27, Theorem 5.3], we deduce that all primes in $\pi(S) \setminus \{p\}$ induce a complete subgraph of $\Delta(S)$ and so also of $\Delta(G)$. By the same reasoning as in Case 2, we can get a contradiction.

Suppose that $S \cong \mathrm{U}_3(q)$ with $q = p^f$ and $|\pi(S)| \leq 5$. If $3 \leq |\pi(S)| \leq 4$, then by Atlas [4], we have that $\pi(S) = \pi(G/N)$, and so $|\Omega| \geq 4$, a contradiction by Lemma 4.7. If $|\pi(S)| = 5$, then by [27, Theorem 5.5], we obtain that all primes in $\pi(S) \setminus \{p\}$ induce a complete subgraph of $\Delta(S)$ and so also of $\Delta(G)$. By the same reasoning as in Case 2, we can get a contradiction.

The proof is complete. □

Now assume that S is isomorphic to one of the groups listed in Lemma 2.11 (3). We will consider the cases $p = 2$ and p is odd separately. We will need Dickson's classification of the subgroups of $\mathrm{L}_2(q)$, and we list them below for convenience.

Lemma 4.13 (see [6, Hauptsatz II.8.27]). *The group $\mathrm{L}_2(p^f)$ has only the following subgroups:*

(1) *elementary abelian p-groups;*

(2) *cyclic groups of order z with $z \mid (p^f \pm 1)/k$, where $k = (p^f - 1, 2)$;*

(3) *dihedral groups of order $2z$ with z as under* (2)*;*

(4) *the alternating group A_4 for $p > 2$ or $p = 2$ and $f \equiv 0 \pmod 2$;*

(5) *the symmetric group S_4 for $p^{2f} - 1 \equiv 0 \pmod{16}$;*

(6) *the alternating group A_5 for $p = 5$ or $p^{2f} - 1 \equiv 0 \pmod 5$;*

(7) *semidirect products of elementary abelian groups of order p^m with cyclic groups of order t, where $t \mid (p^m - 1)/(p^f - 1, 2)$ and $t \mid (p^f - 1)$;*

(8) *$\mathrm{L}_2(p^m)$ for $m \mid f$ and $\mathrm{PGL}_2(p^m)$ for $2m \mid f$.*

The main approach used to prove Lemmas 4.14 and 4.16 in each case is similar, but we will give some details in each case for completeness and convenience. Hence the proofs will not be too long.

Lemma 4.14. *Under Hypothesis 4.5, if S is isomorphic to $\mathrm{L}_2(q)$ with $q = p^f$ and $|\pi(q \pm 1)| \le 4$, where p is an odd prime, then such a G does not exist.*

Proof. By Corollary 4.8, we have that $4 \le |\pi(S)| \le 8$. We will complete the proof one by one.

Case 1: $|\pi(S)| = 4$. By [8, Theorem 3.2 and Lemma 3.4] and Atlas [4], we obtain that $|\pi(G/N)| = 4$ or $|\pi(G/N)| = 5$. By Lemma 4.7, the former case cannot occur. So we may assume that $\pi(G/N) = \pi(S) \cup \{r\}$ and $\Omega = \{p_1, p_2, p_3\}$. By Lemmas 2.12 and 4.7, without loss of generality, we may assume that $\{p_1, p_2\}$ is the only edge in Ω. Since neither p_1 nor p_2 is adjacent to p_3 in $\Delta(G)$, we deduce that p_3 is adjacent to all primes in $\rho(G) \setminus \{p_1, p_2\}$.

If $q - 1$ or $q + 1$ is a power of 2, then it suffices to show that $\pi(q - 1) = \{2\}$ and $\pi(q + 1) = \{2, s, t\}$ since, by the same reasoning, we can also get a contradiction for $|\pi(q - 1)| = 3$ and $|\pi(q + 1)| = 1$. By [27, Theorem 5.2], we have that $\{2, s, t\}$ induces a triangle. It follows from Lemma 2.9 that r is adjacent to all primes in $\{2, s, t\}$. Thus $\{2, s, t, r\}$ induces a complete subgraph K_4, and so it follows that $\{2, s, t, r, p_3\}$ induces a complete subgraph K_5, a contradiction.

If $\pi(q-1) = \{2, s\}$ and $\pi(q+1) = \{2, t\}$, then by [27, Theorem 5.2], we obtain that $\{2, s, t\}$ induces a path with $\deg(2) = 2$. By Lemma 2.9, we obtain that r is adjacent to all primes in $\{2, s, t\}$. Thus $\Delta(G)$ has a subgraph isomorphic to two K_4's sharing three common vertices, a contradiction.

Case 2: $|\pi(S)| = 5$.

Subcase 2.1: $\pi(G/N) = \pi(S)$, and so $|\Omega| = 3$. Write $\Omega = \{p_1, p_2, p_3\}$. By Lemmas 2.12 and 4.7, without loss of generality, we may assume that $\{p_1, p_2\}$ is the only edge in Ω. Since neither p_1 nor p_2 is adjacent to p_3 in $\Delta(G)$, we obtain that p_3 is adjacent to all primes in $\rho(G) \setminus \{p_1, p_2\}$. Let $\psi \in \mathrm{Irr}(N)$ such that $p_1 p_2 \mid \psi(1)$ and $I = I_M(\psi)$. By Clifford Theory, we have that $|M : I|\psi(1)$ divides all of the degrees in $\mathrm{cd}(\psi^M)$.

If $q-1$ or $q+1$ is a power of 2, then it suffices to show that $\pi(q-1) = \{2\}$ and $\pi(q+1) = \{2, r, s, t\}$ since, by the same reasoning, we can also get a contradiction for $|\pi(q-1)| = 4$ and $|\pi(q+1)| = 1$. It follows from [27, Theorem 5.2] that $\{2, r, s, t\}$ induces a complete subgraph K_4. Thus it follows that $\{2, r, s, t, p_3\}$ induces a complete subgraph K_5, a contradiction.

If $|\pi(q+\epsilon)| = 2$ and $|\pi(q-\epsilon)| = 3$ for $\epsilon \in \{1, -1\}$, then it suffices to show that $\pi(q-1) = \{2, r, s\}$ and $\pi(q+1) = \{2, t\}$ since, by the same reasoning, we can also get a contradiction for $|\pi(q-1)| = 2$ and $|\pi(q+1)| = 3$. By [27, Theorem 5.2], we know that $\{2, r, s\}$ induces a triangle and 2 is adjacent to t. Thus $\{2, r, s, p_3\}$ induces a complete subgraph K_4, and this implies that $\{t, p, p_1, p_2\}$ induces a complete subgraph K_4. If $I = M$ and ψ extends to M, then by Lemma 2.6, we obtain that $\psi(1)(q-1)$ and $\psi(1)(q+1)$ belong to $\mathrm{cd}(\psi^M)$. If $I = M$ and ψ does not extend to M, then by [15, Lemma 3.1], we deduce that $\psi(1)(q-1)$ and $\psi(1)(q+1)$ belong to $\mathrm{cd}(\psi^M)$. In both cases, $\Delta(G)$ has a subgraph isomorphic to K_5, a contradiction. So we may assume that $I \lneqq M$. We consider the various possibilities for the subgroup I/N of $M/N \cong S$.

- If I/N is one of the groups listed in Lemma 4.13 (1), (2), (4) and (5), then $|M : I|$ is divisible by at least three distinct primes in $\pi(S)$. Hence $\Delta(G)$ has a subgraph isomorphic to K_5, a contradiction.

- If $I/N = T/N \rtimes H/N$ is one of the groups listed in Lemma 4.13 (3), then $|M : I|$ is divisible by at least two distinct primes in $\pi(S) \setminus \{2\}$. Since T/N is cyclic, we deduce that ψ can extend to T, and we let $\psi_0 \in \mathrm{Irr}(T)$ be such that $(\psi_0)_N = \psi$. If ψ_0 is I-invariant, then ψ_0 and hence ψ extend to I, and so $2\psi(1) \in \mathrm{cd}(\psi^I)$ by Lemma 2.6. If ψ_0 is not I-invariant, then T must be the inertia group of ψ_0 in I since $|I : T| = 2$. It follows that $(\psi_0)^I$ is irreducible, and so $2\psi(1) \in \mathrm{cd}(\psi^I)$. Therefore, we always have $2\psi(1) \in \mathrm{cd}(\psi^I)$. By Clifford Theory, we obtain that $2\psi(1)|M : I|$ divides some degree in $\mathrm{cd}(\psi^M)$. Hence $\Delta(G)$ has a subgraph isomorphic to K_5, a contradiction.

- If I/N is the group listed in Lemma 4.13 (6), that is, $I/N \cong A_5$, then $|M : I|$ is divisible by primes in $\pi(S) \setminus \{2, 3, 5\}$. If ψ extends to I, then by Lemma 2.6, we have that 2 divides some degree in $\mathrm{cd}(\psi^I)$. If ψ does not extend to I, then by [15, Lemma 3.1], we obtain that 2 divides some degree in $\mathrm{cd}(\psi^I)$. In both cases, we deduce that $2|M : I|\psi(1)$ divides some degree in $\mathrm{cd}(\psi^M)$ by Clifford Theory. Hence $\Delta(G)$ has a subgraph isomorphic to K_5, a contradiction.

- If I/N is one of the groups listed in Lemma 4.13 (7), then $2t \mid |M : I|$. Thus $\{2, t, p_1, p_2\}$ induces a complete subgraph K_4, and so $\Delta(G)$ has a subgraph isomorphic to two K_4's sharing a common vertex, a contradiction.

- If I/N is one of the groups listed in Lemma 4.13 (8). By [15, Lemma 3.7], we have that $\{p, p_1, p_2\}$ and one of the primes in $\{2, r, s\}$ induce a complete subgraph K_4, a contradiction.

Subcase 2.2: $\pi(G/N) \setminus \pi(S) \neq \emptyset$. If $q - 1$ or $q + 1$ is a power of 2, then by [27, Theorem 5.2], we deduce that $\pi(S) \setminus \{p\}$ induces a complete subgraph K_4 in $\Delta(S)$. By Lemma 2.9, we know that $\Delta(G/N)$ contains a complete subgraph K_5, a contradiction.

If $|\pi(q + \epsilon)| = 2$ and $|\pi(q - \epsilon)| = 3$ for $\epsilon \in \{1, -1\}$, then it suffices to show that $\pi(q + 1) = \{2, r\}$ and $\pi(q - 1) = \{2, s, t\}$ since, by the same reasoning, we can also get a contradiction for $|\pi(q - 1)| = 2$ and $|\pi(q + 1)| = 3$. By [27, Theorem 5.2], $\{2, s, t\}$ induces a triangle, and r is adjacent to 2 in $\Delta(S)$. If u_1 and u_2 are two distinct primes in $\pi(G/N) \setminus \pi(S)$, then by Lemmas 2.9 and 2.10, we know that $\{2, s, t, u_1, u_2\}$ induces a complete subgraph K_5, a contradiction. Hence $\pi(G/N) = \pi(S) \cup \{u\}$ for some prime $u \in \rho(G)$, and thereby $\Omega = \{p_1, p_2\}$. By Lemma 2.9, we obtain that u is adjacent to all primes in $\{2, r, s, t\}$. Thus both $\{2, s, t, u\}$ and $\{p_2, p_1, p, r\}$ induce K_4's of $\Delta(G)$. Let $\psi \in \mathrm{Irr}(N)$ such that $p_1 \mid \psi(1)$ and $I = I_M(\psi)$. By Clifford Theory, we know that $|M : I|\psi(1)$ divides all of the degrees in $\mathrm{cd}(\psi^M)$. If $I = M$, then $\psi(1)(q - 1)$ and $\psi(1)(q + 1)$ belong to $\mathrm{cd}(\psi^M)$. Hence $\Delta(G)$ has a subgraph isomorphic to two K_4's sharing three common vertices, a contradiction. Assume that $I \lneqq M$. The following proofs are the same as in Subcase 2.1, so we give only the main contradictions.

- If I/N is one of the groups listed in Lemma 4.13 (1), then $\Delta(G)$ has a subgraph isomorphic to K_5, a contradiction.

- If I/N is one of the groups listed in Lemma 4.13 (2) and (3), then we get $\deg(2) \geq 6$, a contradiction.

- If I/N is one of the groups listed in Lemma 4.13 (4) and (5), then $|M : I|$ is divisible by primes in $\pi(S) \setminus \{2, 3\}$. If ψ extends to I, then by Lemma 2.6, we have that $2\psi(1)$ or $3\psi(1)$ divides some degrees in $\mathrm{cd}(\psi^I)$. If ψ does not extend to I, then $2\psi(1)$ or $3\psi(1)$ divides some degree in $\mathrm{cd}(\psi^I)$ by Lemma 2.4. By Clifford Theory, in both cases, we have that $2|M : I|\psi(1)$ or $3|M : I|\psi(1)$ divides some degree in $\mathrm{cd}(\psi^M)$. Thus $\Delta(G)$ has a subgraph isomorphic to K_5, a contradiction.

- If I/N is the group listed in Lemma 4.13 (6), that is, $I/N \cong \mathrm{A}_5$, then $\Delta(G)$ has a subgraph isomorphic to two K_4's sharing a common vertex, a contradiction.
- If $I/N = P/N \rtimes H/N$ is one of the groups listed in Lemma 4.13 (7), then $2r \mid |M : I|$. We have that either s or t divides $|I : P|$; otherwise, $\Delta(G)$ has a subgraph isomorphic to K_5, a contradiction. If ψ does not extend to P, then the character degrees in $\mathrm{cd}(\psi^P)$ are all divisible by p. This implies that the character degrees in $\mathrm{cd}(\psi^I)$ and hence in $\mathrm{cd}(\psi^M)$ are all divisible by p. Thus we have that $p|M : I|\psi(1)$ divides some degree in $\mathrm{cd}(\psi^M)$, and thereby all primes in $\{r, s, t, p, u, p_1\}$ are neighbors of 2, and so $\deg(2) \geq 6$, a contradiction. So we may assume that ψ extends to P. Since H/N is cyclic, by [9, Corollary 11.31], we know that ψ extends to I. By Lemma 2.6, we obtain that $|I : P|\psi(1)$ divides some degree in $\mathrm{cd}(\psi^I)$, and so, using Clifford Theory, we obtain that $|M : I||I : P|\psi(1)$ divides some degree in $\mathrm{cd}(\psi^M)$, and thereby $\Delta(G)$ has a subgraph isomorphic to two K_4's sharing two common vertices, a contradiction.
- If I/N is one of the groups listed in Lemma 4.13 (8), then it follows from [15, Lemma 3.7] that $\pi(|M : I|)$ has nontrivial intersection with each of the three sets $\{p\}$, $\pi(q-1)$ and $\pi(q+1)$, and we get a contradiction.

By the same reasoning as in Case 2, we can get contradictions for $6 \leq |\pi(S)| \leq 7$, so the proof is omitted. If $|\pi(S)| = 8$, then by [27, Theorem 5.2], we obtain that $\Delta(G)$ has a subgraph isomorphic to two K_4's sharing a common vertex, a contradiction.

The proof is complete. □

Lemma 4.15. *Suppose that $\pi(2^f - 1) \cup \pi(2^f + 1) = \{3, p_1, p_2, r\}$ for an integer $f \geq 1$. If $|\pi(2^f - \epsilon)| = 1$ and $|\pi(2^f + \epsilon)| = 3$ for $\epsilon \in \{1, -1\}$, then $\epsilon = 1$ and either*

(1) *$f \geq 11$ is a prime, $2^f + 1 = 3^a \cdot p_1^b \cdot p_2^c$ with $a, b, c \geq 1$ and $2^f - 1 = r$ is a Mersenne prime; or*

(2) *$f = 8$, $2^f + 1 = 257$ and $2^f - 1 = 3 \cdot 5 \cdot 17$.*

Proof. It is obvious that $f \geq 8$. If f is odd, then $2^f + 1 \equiv 0 \pmod 3$. By [8, Lemma 1.2], we deduce that $2^f + 1$ is not a power of 3. Therefore, assume without loss of generality that $2^f + 1 = 3^a \cdot p_1^b \cdot p_2^c$ with $a, b, c \geq 1$ and $2^f - 1 = r^d$.

By [8, Lemma 1.2] again, we have that $d = 1$ and $r = 2^f - 1$ is a Mersenne prime; in particular, $f \geq 11$ is a prime. This is situation (1). If f is even, then $2^f - 1 \equiv 0 \pmod 3$. By [8, Lemma 1.2], we have that $2^f - 1$ is not a power of 3. Therefore, assume without loss of generality that $2^f - 1 = 3^a \cdot p_1^b \cdot p_2^c$ with $a, b, c \geq 1$ and $2^f + 1 = r^d$. By [8, Lemma 1.2] again, we deduce that $d = 1$ and $r = 2^f + 1$ is a Fermat prime; in particular, $f = 2^n \geq 8$, and so $f = 4m$. Thus

$$2^f - 1 = 3^a \cdot p_1^b \cdot p_2^c = (2^m - 1)(2^m + 1)(2^{2m} + 1).$$

Observe that $(2^m - 1, 2^m + 1) = (2^{2m} - 1, 2^{2m} + 1) = 1$. By [8, Lemma 1.2], we must have $m = 2$, and so $2^f + 1 = 257$ and $2^f - 1 = 3 \cdot 5 \cdot 17$. This is situation (2). The proof is complete. □

We will frequently use the fact that, if $S \cong \mathrm{L}_2(2^f)$, then $\Delta(S)$ has three connected components $\{2\}$, $\pi(2^f - 1)$ and $\pi(2^f + 1)$, and each component is a complete graph, which can be found in [27, Theorem 5.2]. We will also make use of the facts that the Schur multiplier for $\mathrm{L}_2(2^f)$ has order 1, where $2^f > 4$, and the only outer automorphism of $\mathrm{L}_2(2^f)$ is the field automorphism. Notice that the group listed in Lemma 4.13 (5) cannot be a subgroup of $\mathrm{L}_2(2^f)$.

Lemma 4.16. *Under Hypothesis 4.5, if S is isomorphic to*

$$\mathrm{L}_2(2^f) \quad \textit{with } |\pi(2^f \pm 1)| \leq 4,$$

then such a G does not exist.

Proof. By Corollary 4.8, we have that $4 \leq |\pi(S)| \leq 8$. We will complete the proof case by case.

Case 1: $|\pi(S)| = 4$. By [8, Lemma 3.5], we have that either $S \cong \mathrm{L}_2(2^4)$ or $f \geq 5$ is prime, $2^f - 1 = r$ is prime, and $2^f + 1 = 3 \cdot t^\beta$ with t an odd prime and $\beta \geq 1$ odd. By Atlas [4], we have that $|\pi(G/N)| = 4$ or $|\pi(G/N)| = 5$. It follows from Lemma 4.7 that the former case cannot occur. So we may assume that $|G/N : M/N| = f$, and so $\pi(G/N) = \pi(S) \cup \{f\}$ and $\Omega = \{p_1, p_2, p_3\}$. By Lemmas 2.12 and 4.7, without loss of generality, we may assume that $\{p_1, p_2\}$ is the only edge in Ω. Since neither p_1 nor p_2 is adjacent to p_3 in $\Delta(G)$, we deduce that p_3 is adjacent to all primes in $\{2, 3, t, r, f\}$. By Lemma 2.9, we obtain that f is adjacent to all primes in $\{3, t, r\}$. By [27, Theorem 5.2], we know that 3 is adjacent to t. From the structure of $\Delta(G)$, without loss of generality, we may assume that p_1 is adjacent to 3 and p_2 is adjacent to t. Let $\chi_1, \chi_2 \in \mathrm{Irr}(G)$ such that $3p_1 \mid \chi_1(1)$ and $tp_2 \mid \chi_2(1)$, and $\psi_i \in \mathrm{Irr}(M)$ an irreducible constituent of $(\chi_i)_M$ for $i = 1, 2$. Then $\chi_i(1)/\psi_i(1) \mid |G/M| = f$ by Lemma 2.4, and so $3p_1 \mid \psi_1(1)$

and $tp_2 \mid \psi_2(1)$. Since $\Delta(G)$ has no K_4 containing both 3 and p_1, we deduce that $|\pi(\psi_1(1)) \setminus \{3, p_1\}| \leq 1$. Similarly, $|\pi(\psi_2(1)) \setminus \{t, p_2\}| \leq 1$. Let θ_1 and θ_2 be irreducible constituents of $(\psi_1)_N$ and $(\psi_2)_N$, respectively, and $I_i = I_M(\theta_i)$ for $i = 1, 2$. We have that $|\pi(|M : I_1|) \setminus \{3, p_1\}| \leq 1$ since $|\pi(\psi_1(1)) \setminus \{3, p_1\}| \leq 1$.

Assume that $I_1 \lneqq M$. We consider the various possibilities for the subgroup I_1/N of $M/N \cong S$. Since $f \geq 5$ is a prime, we deduce that I_1/N cannot be the group listed in Lemma 4.13 (4).

- If I_1/N is one of the groups listed in Lemma 4.13 (1)–(3), (6) and (8), then we have that $|\pi(|M : I_1|) \setminus \{3, p_1\}| \geq 2$, a contradiction.
- If $I_1/N = P/N \rtimes H/N$ is one of the groups listed in Lemma 4.13 (7), then $3t \mid |M : I_1| \mid \psi_1(1)$. We have that $r \mid |I_1 : P|$; otherwise, $r \mid |M : I_1|$, a contradiction. If θ_1 does not extend to P, then the character degrees in $\mathrm{cd}((\theta_1)^P)$ are all divisible by 2. This implies that the character degrees in $\mathrm{cd}((\theta_1)^I)$ and hence in $\mathrm{cd}((\theta_1)^M)$ are all divisible by 2. Thus $2|M : I_1|\theta_1(1) \mid \psi_1(1)$, and thereby $\Delta(G)$ has at least two K_4's sharing two common vertices, a contradiction. Hence we may assume that θ_1 extends to P. Since H/N is cyclic, by [9, Corollary 11.31], we have that θ_1 extends to I_1. By Lemma 2.6, we obtain that $|I_1 : P|\theta_1(1)$ divides some degree in $\mathrm{cd}((\theta_1)^I)$, and so, by Clifford Theory, we know that $|M : I_1||I_1 : P|\theta_1(1)$ divides some degree in $\mathrm{cd}((\theta_1)^M)$, and thereby $\Delta(G)$ has at least two K_4's sharing two common vertices, a contradiction.

Hence we may assume that $I_1 = M$. Similarly, $I_2 = M$. Since the Schur multiplier of $M/N \cong \mathrm{L}_2(2^f)$ with $f \geq 5$ is trivial, by [9, Theorem 11.7], we obtain that θ_1 extends to $\hat{\theta}_1 \in \mathrm{Irr}(M)$. By Lemma 2.6, we have that

$$\psi_1(1) \in \{\theta_1(1), 2^f\theta_1(1), (2^f - 1)\theta_1(1), (2^f + 1)\theta_1(1)\} \subseteq \mathrm{cd}(M).$$

In all cases, we must have $p_1 \mid \theta_1(1)$. Since $(2^f + 1)\theta_1(1) \in \mathrm{cd}(M)$, we deduce that $\{3, t, p_1\}$ induces a triangle. Similarly, $\{3, t, p_2\}$ induces a triangle. Hence $\{3, t, p_1, p_2\}$ induces a complete subgraph K_4, and so $\Delta(G)$ has at least two K_4's sharing two common vertices, a contradiction.

Case 2: $|\pi(S)| = 5$, and so $f \geq 6$.

Subcase 2.1: $|\pi(2^f - \epsilon)| = 1$, $|\pi(2^f + \epsilon)| = 3$ for $\epsilon \in \{1, -1\}$. If $S \ncong \mathrm{L}_2(2^8)$, then by Lemma 4.15, we have that $f \geq 11$ is a prime, $2^f + 1 = 3^a \cdot p_1^b \cdot p_2^c$ with $a, b, c \geq 1$ and $2^f - 1 = r$ is a Mersenne prime. If $|G/N : M/N| = f$, then by Lemma 2.9, we obtain that f is adjacent to all primes in $\{3, p_1, p_2, r\}$, and so, by [27, Theorem 5.2], we know that $\{3, p_1, p_2, f\}$ induces a complete subgraph K_4. If $G/N = M/N$, then $\Omega = \{p_3, p_4, p_5\}$. By Lemmas 2.12 and 4.7, without loss of generality, we may assume that $\{p_3, p_4\}$ is the only edge in Ω. Since neither p_3 nor p_4 is adjacent to p_5 in $\Delta(G)$, we have that p_5 is adjacent to all primes in $\{2, 3, p_1, p_2, r\}$. By [27, Theorem 5.2], we obtain that $\{3, p_1, p_2, p_5\}$ induces a complete subgraph K_4. In both cases, we can get a contradiction using the same reasoning as in Case 1.

Hence we may assume that $S \cong \mathrm{L}_2(2^8)$, and so $\Omega = \{q_1, q_2, q_3\}$. By Lemmas 2.12 and 4.7, without loss of generality, we may assume that $\{q_1, q_2\}$ is the only edge in Ω. Since neither q_1 nor q_2 is adjacent to q_3 in $\Delta(G)$, we deduce that q_3 is adjacent to all primes in $\{2, 3, 5, 17, 257\}$. It follows from [27, Theorem 5.2] that $\{3, 5, 17, q_3\}$ induces a complete subgraph K_4. By similar reasoning to Case 1, we can get a contradiction.

Subcase 2.2: $\pi(2^f - 1) = \{p_1, p_2\}$ and $\pi(2^f + 1) = \{p_3, p_4\}$. By Lemmas 2.9 and 2.10, we have that $|\pi(G/N) \setminus \pi(M/N)| \le 1$. Let $|G/N : M/N| = t$. By [28, Theorem A], we deduce that $t(2^f \pm 1) \in \mathrm{cd}(G/N)$.

(1) $\pi(G/N) = \pi(M/N) \cup \{r\}$ and $\Omega = \{s, p\}$. Then at most one prime in $\{p_i\}_{i=1}^4$ is in $\pi(t)$, say p_1. We have that $2 \notin \pi(t)$; otherwise, $\Delta(G)$ has two K_4's sharing two common vertices, a contradiction. From the structure of $\Delta(G)$, we obtain that r together with at least two primes in $\{p_i\}_{i=1}^4$ are contained in the same complete subgraph K_4.

If $\pi(t) = \{r, p_1\}$, then $\{p_1, r, p_3, p_4\}$ induces a complete subgraph K_4 and so does $\{p_2, 2, p, s\}$. Thus there exists $\chi \in \mathrm{Irr}(G)$ such that $ps \mid \chi(1)$. Let $\psi \in \mathrm{Irr}(N)$ be an irreducible constituent of χ_N and $I = I_M(\psi)$. Then $\chi(1)/\psi(1) \mid |G/N|$ by Lemma 2.4, and so $ps \mid \psi(1)$. Similar to Case 1, if $I \lneqq M$, then I/N cannot be one of the groups listed in Lemma 4.13 (1)–(7). If I/N is one of the groups listed in Lemma 4.13 (8), then by [15, Lemma 3.7], we have that $|M : I|$ has nontrivial intersection with each of the three sets $\{2\}$, $\pi(2^f - 1)$ and $\pi(2^f + 1)$. Thus $|M : I|$ is divisible by at least three primes in $\pi(S)$, and so $\Delta(G)$ would contain a subgraph isomorphic to K_5, a contradiction. If $I = M$, then we can get a contradiction using the same reasoning as in Case 1.

If $\pi(t) = \{r\}$, then we consider the various possibilities for r together with at least two primes in $\{p_i\}_{i=1}^4$ are contained in the same complete subgraph K_4.

- Assume without loss of generality that $\{p_1, p_2, p_3, r\}$ induces a complete subgraph K_4. Using the same reasoning as above, we can get a contradiction.
- Assume without loss of generality that $\{p_1, p_2, 2, r\}$ induces a complete subgraph K_4. From the structure of $\Delta(G)$, we know that p or s, say p, is adjacent to p_1 or p_2, say p_1, and p is not adjacent to p_2. Let $\chi \in \mathrm{Irr}(G)$ such that $pp_1 \mid \chi(1)$ and $\psi \in \mathrm{Irr}(M)$ an irreducible constituent of χ_M. Then we have that $\chi(1)/\psi(1) \mid |G/M| = t$ by Lemma 2.4, and so $pp_1 \mid \psi(1)$. Since $\Delta(G)$ has no K_4 containing both p and p_1, we have that $|\pi(\psi(1)) \setminus \{p, p_1\}| \le 1$. Let $\theta \in \mathrm{Irr}(N)$ be an irreducible constituent of ψ_N and $I = I_M(\theta)$. Assume that $I = M$. Since the Schur multiplier of $M/N \cong \mathrm{L}_2(2^f)$ with $f \ge 6$ is trivial, by

[9, Theorem 11.7], we obtain that θ extends to $\theta_0 \in \mathrm{Irr}(M)$. By Lemma 2.6, we have that

$$\psi(1) \in \{\theta(1), 2^f\theta(1), (2^f-1)\theta(1), (2^f+1)\theta(1)\} \subseteq \mathrm{cd}(M).$$

In all cases, we must have $p \mid \theta(1)$. Since $(2^f-1)\theta(1) \in \mathrm{cd}(M)$, we deduce that p is adjacent to p_2, a contradiction. Hence we may assume that $I \lneqq M$. By the same reasoning as in Case 1, I/N cannot be one of the groups listed in Lemma 4.13 (1)–(7). If I/N is one of the groups listed in Lemma 4.13 (8), then by [15, Lemma 3.7], we have that $|M : I|$ is divisible by at least three primes in $\pi(S)$, and so $\Delta(G)$ would contain a subgraph isomorphic to two K_4's sharing three common vertices, a contradiction.

- Assume without loss of generality that $\{p_1, p_2, s, r\}$ induces a complete subgraph K_4. If p is adjacent to s, then there exists $\chi \in \mathrm{Irr}(G)$ such that $ps \mid \chi(1)$. Let $\psi \in \mathrm{Irr}(N)$ be an irreducible constituent of χ_N. Then $\chi(1)/\psi(1) \mid |G/N|$ by Lemma 2.4, and so $ps \mid \psi(1)$. By Lemma 2.5, we have that p, s and two of the primes in $\{2\} \cup \{p_i\}_{i=1}^4$ are contained in the same complete subgraph K_4, a contradiction. So p is not adjacent to s, and thereby p must be adjacent to both p_1 and p_2. Thus s is adjacent to p_3 or p_4, say p_3, and s is not adjacent to p_4. By similar reasoning to above, we can get a contradiction.
- Assume without loss of generality that $\{p_1, p_3, r, s\}$ induces a complete subgraph K_4. Using the same reasoning as above, we have that p is not adjacent to s. Thus there is no way to guarantee that $\Delta(G)$ is isomorphic to the graph depicted in Figure 2, a contradiction.
- Assume without loss of generality that $\{p_1, p_3, r, 2\}$ induces a complete subgraph K_4. Then p is adjacent to 2, and by similar reasoning to above, we can get a contradiction.

(2) $\pi(G/N) = \pi(M/N)$ and $\Omega = \{r, s, t\}$. By Lemmas 2.12 and 4.7, without loss of generality, we may assume that $\{r, s\}$ is the only edge in Ω. Since neither r nor s is adjacent to t in $\Delta(G)$, we obtain that the prime t is adjacent to all primes in $\{2, p_1, p_2, p_3, p_4\}$. By the same reasoning as in (1), we can get a contradiction.

Case 3: $|\pi(S)| = 6$, and so $f \geq 10$.

Subcase 3.1: $|\pi(2^f + \epsilon)| = 1$ and $|\pi(2^f - \epsilon)| = 4$ for $\epsilon \in \{1, -1\}$. It suffices to show that $\pi(2^f - 1) = \{p_1, p_2, p_3, p_4\}$ and $\pi(2^f + 1) = \{p_5\}$ since, by the same reasoning, we can also get a contradiction for

$$|\pi(2^f - 1)| = 1 \quad \text{and} \quad |\pi(2^f + 1)| = 4.$$

It follows from Lemma 2.9 and [28, Theorem A] that $\pi(S) = \pi(G/N)$. Let

$$|G/N : M/N| = t \quad \text{and} \quad \rho(G) = \pi(S) \cup \{p, r\}.$$

By [28, Theorem A], we know that $t(2^f \pm 1) \in \mathrm{cd}(G/N)$, and so it follows that $\pi(t) \subseteq \{p_1, p_2, p_3, p_4\}$ and $|\pi(t)| \leq 2$. From the structure of $\Delta(G)$, there exists some prime in $\{p_1, p_2, p_3, p_4\} \setminus \pi(t)$, say p_1, such that p_1 is adjacent to p or r, say p, and p_1 is not adjacent to p_5. Let $\chi \in \mathrm{Irr}(G)$ such that $pp_1 \mid \chi(1)$ and $\psi \in \mathrm{Irr}(M)$ an irreducible constituent of χ_M. Then $\chi(1)/\psi(1) \mid |G/M| = t$ by Lemma 2.4, and so $pp_1 \mid \psi(1)$. Since $\Delta(G)$ has no K_4 containing both p and p_1, we have that $|\pi(\psi(1)) \setminus \{p, p_1\}| \leq 1$. Let $\theta \in \mathrm{Irr}(N)$ be an irreducible constituent of ψ_N and $I = I_M(\theta)$. By the same reasoning as in Subcase 2.2, we can get a contradiction.

Subcase 3.2: $|\pi(2^f + \epsilon)| = 2$ and $|\pi(2^f - \epsilon)| = 3$ for $\epsilon \in \{1, -1\}$. It suffices to show that $\pi(2^f - 1) = \{p_1, p_2, p_3\}$ and $\pi(2^f + 1) = \{p_4, p_5\}$ since, by the same reasoning, we can also get a contradiction for

$$|\pi(2^f - 1)| = 2 \quad \text{and} \quad |\pi(2^f + 1)| = 3.$$

Let $|G/N : M/N| = t$ and $\rho(G) = \pi(S) \cup \{p, r\}$. By [28, Theorem A], we obtain that $t(2^f \pm 1) \in \mathrm{cd}(G/N)$.

If $\pi(G/N) \neq \pi(M/N)$, then $|\pi(t)| = 1$, say $\pi(t) = \{p\}$. From the structure of $\Delta(G)$, there exists a prime in $\{p_1, p_2, p_3\}$, say p_1, such that p_1 is adjacent to r, and $\Delta(G)$ has no K_4 containing both r and p_1. By the same reasoning as in Subcase 2.2, we can get a contradiction.

So we may assume that $\pi(G/N) = \pi(M/N)$. If $2 \in \pi(t)$, then $t = 2^\alpha$ with $\alpha \geq 1$ an integer. Thus there exists a prime in $\{p_1, p_2, p_3\}$, say p_1, such that p_1 is adjacent to r, and $\Delta(G)$ has no K_4 containing both p_1 and r. By the same reasoning as in Subcase 2.2, we can get a contradiction.

Now we assume that $2 \notin \pi(t)$. Since $t(2^f \pm 1) \in \mathrm{cd}(G/N)$, we deduce that $\pi(t)$ has at most two primes which are contained in $\{p_i\}_{i=1}^5$, and p_4 and p_5 cannot be contained in $\pi(t)$ together. If p_4 or p_5 is in $\pi(t)$, say p_4, then there exists a prime in $\{p_i\}_{i=1}^3 \setminus \pi(t)$, say p_1, such that p_1 is adjacent to p or r, say p, and $\Delta(G)$ has no K_4 containing both p and p_1. By the same reasoning as in Subcase 2.2, we can get a contradiction. If two of the primes in $\{p_1, p_2, p_3\}$ are in $\pi(t)$, say p_1 and p_2, then there exists a prime p or r, say p, such that p is adjacent to p_4 or p_5, say p_4, and p is not adjacent to p_5. By the same reasoning as in Subcase 2.2, we can get a contradiction.

Hence, without loss of generality, we have that $t = p_1^\alpha$ with $\alpha \geq 0$ being an integer. If $\alpha \neq 0$, then using the same reasoning as in Subcase 2.2, we can get that p and p_2 would be contained in the same complete subgraph K_4. Similarly, p and p_3 would be contained in the same complete subgraph K_4, r and p_2 would be contained in the same complete subgraph K_4, r and p_3 would be contained in the same complete subgraph K_4. Hence $\{p_2, p_3, p, r\}$ induces a complete subgraph K_4.

Assume without loss of generality that p is adjacent to p_4 (this assumption is reasonable since p must be adjacent to at least one of the primes in $\{p_4, p_5\}$). Let $\chi \in \mathrm{Irr}(G)$ such that $pp_4 \mid \chi(1)$, and $\psi \in \mathrm{Irr}(M)$ an irreducible constituent of χ_M. Then $\chi(1)/\psi(1) \mid |G/M| = p_1^{\alpha}$ by Lemma 2.4, and so $pp_4 \mid \psi(1)$. Since $\Delta(G)$ has no K_4 containing both p and p_4, we have that $|\pi(\psi(1)) \setminus \{p, p_4\}| \leq 1$. Let $\theta \in \mathrm{Irr}(N)$ be an irreducible constituent of ψ_N and $I = I_M(\theta)$. Similar to Subcase 2.2, we can get a contradiction. Hence we may assume that $\alpha = 0$, that is, $G/N = M/N \cong S$. Using a similar argument to above, we can get that every prime in $\{p_1, p_2, p_3\}$ is adjacent to at most one prime in $\{p_4, p_5\}$, and every prime in $\{p_4, p_5\}$ is adjacent to at most one prime in $\{p_1, p_2, p_3\}$. Clearly, the case that there exists at most one edge between $\{p_1, p_2, p_3\}$ and $\{p_4, p_5\}$ cannot occur. Hence we may assume without loss of generality that p_1 is adjacent to p_4, and p_2 is adjacent to p_5. From the structure of $\Delta(G)$, we know that p_1 and at least one prime in $\{p_2, p_3, p_4\}$ are contained in the same complete subgraph K_4. In all cases, we can get a contradiction using the same reasoning as in Subcase 2.2.

Case 4: $|\pi(S)| = 7$, and so $f \geq 12$. From Lemma 2.9, we have that $\pi(S) = \pi(G/N)$. Let $|G/N : M/N| = t$ and $\rho(G) = \pi(S) \cup \{p\}$. By [28, Theorem A], we know that $t(2^f \pm 1) \in \mathrm{cd}(G/N)$.

Subcase 4.1: $|\pi(2^f + \epsilon)| = 2$ and $|\pi(2^f - \epsilon)| = 4$ for $\epsilon \in \{1, -1\}$. It suffices to show that $\pi(2^f - 1) = \{p_1, p_2, p_3, p_4\}$ and $\pi(2^f + 1) = \{p_5, p_6\}$ since, by the same reasoning, we can also get a contradiction for $|\pi(2^f - 1)| = 2$ and $|\pi(2^f + 1)| = 4$. By [27, Theorem 5.2], we have that $\pi(t) \subseteq \{p_1, p_2, p_3, p_4\}$ and $|\pi(t)| \leq 1$. Otherwise, $\Delta(G)$ would contain a subgraph isomorphic to K_5 or $\deg(2) \geq 6$, a contradiction. Assume without loss of generality that $t = p_1^{\alpha}$, where α is a nonnegative integer. By [27, Theorem 5.2], we know that p is adjacent to exactly two primes in $\{p_1, p_2, p_3, p_4\}$, say p_2 and one of the primes in $\{p_1, p_3, p_4\}$. Hence there exists $\chi \in \mathrm{Irr}(G)$ such that $pp_2 \mid \chi(1)$. Let $\psi \in \mathrm{Irr}(M)$ be an irreducible constituent of χ_M. Then $\chi(1)/\psi(1) \mid |G/M| = p_1^{\alpha}$ by Lemma 2.4, and so $pp_2 \mid \psi(1)$. Since $\Delta(G)$ has no K_4 containing both p and p_2, we have that $|\pi(\psi(1)) \setminus \{p, p_2\}| \leq 1$. Let $\theta \in \mathrm{Irr}(N)$ be an irreducible constituent of ψ_N and $I = I_M(\theta)$. Similar to Subcase 2.2, we can get a contradiction.

Subcase 4.2: $\pi(2^f + 1) = \{p_1, p_2, p_3\}$ and $\pi(2^f - 1) = \{p_4, p_5, p_6\}$. By [27, Theorem 5.2], we deduce that both $\{p_1, p_2, p_3\}$ and $\{p_4, p_5, p_6\}$ induce triangles. By the same reasoning as in Subcase 2.2, we may assume that $t = p_1^{\alpha}$, where α is a nonnegative integer. If $\alpha \neq 0$, then all primes in $\{p_2, p_3, p_4, p_5, p_6\}$ are neighbors of p_1, and hence $\{p_1, p_4, p_5, p_6\}$ induces a complete subgraph K_4. From the structure of $\Delta(G)$, we know that the prime p is adjacent to exactly two primes in $\{p_4, p_5, p_6\}$, say p_4 and p_5. Hence there exists $\chi \in \mathrm{Irr}(G)$ such that $pp_4 \mid \chi(1)$. Let $\psi \in \mathrm{Irr}(M)$ be an irreducible constituent of χ_M. Then we have that $\chi(1)/\psi(1) \mid |G/M| = p_1^{\alpha}$ by Lemma 2.4, and so $pp_4 \mid \psi(1)$. Since $\Delta(G)$ has no K_4 containing both p and p_4, we have that $|\pi(\psi(1)) \setminus \{p, p_4\}| \le 1$. Let $\theta \in \mathrm{Irr}(N)$ be an irreducible constituent of ψ_N and $I = I_M(\theta)$. By the same reasoning as in Subcase 2.2, we can get a contradiction. Hence we may assume that $\alpha = 0$, that is, $G/N \cong S$. From the structure of $\Delta(G)$, there exists some p_i in $\{p_j\}_{j=1}^{6}$, say p_1, such that p is adjacent to p_1 in $\Delta(G)$, and $\Delta(G)$ has no K_4 containing both p and p_1. Let $\chi \in \mathrm{Irr}(G)$ such that $pp_1 \mid \chi(1)$. Since $\Delta(G)$ has no K_4 containing both p and p_1, we have that $|\pi(\chi(1)) \setminus \{p, p_1\}| \le 1$. Let $\psi \in \mathrm{Irr}(N)$ be an irreducible constituent of χ_N and $I = I_G(\psi)$. By the same reasoning as in Subcase 2.2, we can get a contradiction.

Case 5: $|\pi(S)| = 8$, and so $f \ge 20$. It follows that $\pi(S) = \pi(G/N) = \rho(G)$. If $G/N \ne M/N$, then by [28, Theorem A], we know that

$$|G/N : M/N|(2^f \pm 1) \in \mathrm{cd}(G/N).$$

Let $r \in \pi(|G : M|)$. Then r is adjacent to all primes in $\pi(2^{2f} - 1) \setminus \{r\}$, and so $\deg(r) \ge 6$, a contradiction. So we may assume that $G/N = M/N \cong S$. It suffices to show that $\pi(2^f - 1) = \{p_1, p_2, p_3\}$ and $\pi(2^f + 1) = \{p_4, p_5, p_6, p_7\}$ since, by the same reasoning, we can also get a contradiction for $|\pi(2^f - 1)| = 4$ and $|\pi(2^f + 1)| = 3$. By [27, Theorem 5.2], we obtain that 2 is adjacent to exactly two primes in $\{p_4, p_5, p_6, p_7\}$, say p_4 and p_5. Hence there exists $\chi \in \mathrm{Irr}(G)$ such that $2p_4 \mid \chi(1)$. Since $\Delta(G)$ has no K_4 containing both 2 and p_4, we know that $|\pi(\chi(1)) \setminus \{2, p_4\}| \le 1$. Let $\psi \in \mathrm{Irr}(N)$ be an irreducible constituent of χ_N and $I = I_G(\psi)$. By the same reasoning as in Subcase 2.2, we can get a contradiction.

The proof is complete. □

Proof of Theorem 1.1. This follows directly from Lemma 4.1, Theorem 3.1 and Theorem 4.2. □

Acknowledgments. We would like to thank Prof. Lihua Feng for his careful reading of this paper and for the helpful suggestions in its writing. We would also like to thank the editor and reviewers for their valuable suggestions and useful comments.

Bibliography

[1] Z. Akhlaghi, S. Dolfi, E. Pacifici and L. Sanus, Bounding the number of vertices in the degree graph of a finite group, *J. Pure Appl. Algebra* **224** (2020), no. 2, 725–731.

[2] Z. Akhlaghi, K. Khedri and B. Taeri, Finite groups with K_5-free prime graphs, *Comm. Algebra* **47** (2019), no. 7, 2577–2603.

[3] J. A. Bondy and U. S. R. Murty, *Graph Theory*, Grad. Texts in Math. 244, Springer, New York, 2008.

[4] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, $\mathbb{ATLAS}$ *of Finite Groups*, Oxford University, Eynsham, 1985.

[5] M. Ebrahimi, M. Khatami and Z. Mirzaei, Regular character-graphs whose eigenvalues are greater than or equal to -2, *Discrete Math.* **346** (2023), no. 1, Paper No. 113137.

[6] B. Huppert, *Endliche Gruppen. I*, Grundlehren Math. Wiss. 134, Springer, Berlin, 1967.

[7] B. Huppert, Research in representation theory at Mainz (1984–1990), in: *Representation Theory of Finite Groups and Finite-Dimensional Algebras* (Bielefeld 1991), Progr. Math. 95, Birkhäuser, Basel (1991), 17–36.

[8] B. Huppert and W. Lempken, Simple groups of order divisible by at most four primes, *Proc. F. Scorina Gomel State Univ.* **16** (2000), no. 3, 64–75.

[9] I. M. Isaacs, *Character Theory of Finite Groups*, Pure Appl. Math. 69, Academic Press, New York, 1976.

[10] A. Jafarzadeh and A. Iranmanesh, On simple K_n-groups for $n = 5, 6$, in: *Groups St. Andrews 2005. Vol. 2*, London Math. Soc. Lecture Note Ser. 340, Cambridge University, Cambridge (2007), 517–526.

[11] M. L. Lewis, Solvable groups whose degree graphs have two connected components, *J. Group Theory* **4** (2001), no. 3, 255–275.

[12] M. L. Lewis and J. K. McVey, Character degree graphs of automorphism groups of characteristically simple groups, *J. Group Theory* **12** (2009), no. 3, 387–391.

[13] M. L. Lewis and D. L. White, Four-vertex degree graphs of nonsolvable groups, *J. Algebra* **378** (2013), 1–11.

[14] O. Manz, R. Staszewski and W. Willems, On the number of components of a graph related to character degrees, *Proc. Amer. Math. Soc.* **103** (1988), no. 1, 31–37.

[15] J. K. McVey, Bounding graph diameters of nonsolvable groups, *J. Algebra* **282** (2004), no. 1, 260–277.

[16] G. O. Michler, A finite simple group of Lie type has p-blocks with different defects, $p \neq 2$, *J. Algebra* **104** (1986), no. 2, 220–230.

[17] W. M. Montanaro, Jr., *Character degree graphs of almost simple groups*, Ph.D. Thesis, Kent State University, 2014.

[18] A. Moretó and P. H. Tiep, Prime divisors of character degrees, *J. Group Theory* **11** (2008), no. 3, 341–356.

[19] C. P. Morresi Zuccari, Regular character degree graphs, *J. Algebra* **411** (2014), 215–224.

[20] P. P. Pálfy, On the character degree graph of solvable groups. I. Three primes, *Period. Math. Hungar.* **36** (1998), no. 1, 61–65.

[21] Z. Sayanjali, Z. Akhlaghi and B. Khosravi, On the regularity of character degree graphs, *Bull. Aust. Math. Soc.* **100** (2019), no. 3, 428–433.

[22] H. P. Tong-Viet, Groups whose prime graphs have no triangles, *J. Algebra* **378** (2013), 196–206.

[23] H. P. Tong-Viet, Finite groups whose prime graphs are regular, *J. Algebra* **397** (2014), 18–31.

[24] D. L. White, Degree graphs of simple groups of exceptional Lie type, *Comm. Algebra* **32** (2004), no. 9, 3641–3649.

[25] D. L. White, Degree graphs of simple linear and unitary groups, *Comm. Algebra* **34** (2006), no. 8, 2907–2921.

[26] D. L. White, Degree graphs of simple orthogonal and symplectic groups, *J. Algebra* **319** (2008), no. 2, 833–845.

[27] D. L. White, Degree graphs of simple groups, *Rocky Mountain J. Math.* **39** (2009), no. 5, 1713–1739.

[28] D. L. White, Character degrees of extensions of $\mathrm{PSL}_2(q)$ and $\mathrm{SL}_2(q)$, *J. Group Theory* **16** (2013), no. 1, 1–33.

[29] R. A. Wilson, *The Finite Simple Groups*, Grad. Texts in Math. 251, Springer, London, 2009.

[30] J. Zhang, On a problem by Huppert, *Acta Sci. Natur. Univ. Pekin.* **34** (1998), no. 2–3, 143–150.

[31] J. Zhang, A note on character degrees of finite solvable groups, *Comm. Algebra* **28** (2000), no. 9, 4249–4258.

Received March 8, 2023; revised April 5, 2023

Author information

Corresponding author:
Weijun Liu, School of Mathematics and Statistics, HNP-LAMA,
Central South University, Changsha, Hunan, 410083;
and College of General Education,
Guangdong University of Science and Technology,
Dongguan, Guangdong, 523083, P. R. China.
E-mail: wjliu6210@126.com

Lu Lu, School of Mathematics and Statistics, HNP-LAMA,
Central South University, Changsha, Hunan, 410083, P. R. China.
E-mail: lulugdmath@163.com

Qinghong Guo, School of Mathematics and Statistics, HNP-LAMA,
Central South University, Changsha, Hunan, 410083, P. R. China.
E-mail: 1806301009@st.gxu.edu.cn