Discrete Mathematics 346 (2023) 113258

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Distance integral generalized wheel graphs

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ARTICLE INFO

Article history:
Received 8 December 2020
Received in revised form 27 September 2021
Accepted 4 November 2022
Available online xxxx

Keywords:
Distance spectrum
Distance integral graph
Generalized wheel graph

ABSTRACT

For positive integers a, m and n, the graph $aK_m \nabla C_n$ is called the generalized wheel graph. A graph is called distance integral if all eigenvalues of its distance matrix are integers. In this paper, we obtain all distance integral generalized wheel graphs, which consist of an infinite class of graphs and 13 scattered graphs.

1. Introduction

All graphs considered in this article are finite, undirected and simple. Let $G = (V, E)$ be a connected graph with vertex set $V = \{v_1, v_2, \dots, v_n\}$ and edge set $E = \{e_1, e_2, \dots, e_m\}$. The *distance* between v_i and v_j, denoted by $d_G(v_i, v_j)$ (or $d(v_i, v_j)$ for short), is defined as the length of a shortest path between them. The *distance matrix* of G, denoted by $\mathcal{D}(G)$, is the $n \times n$ matrix whose (i, j)-entry is equal to $d_G(v_i, v_j)$ for $1 \le i, j \le n$. Since $\mathcal{D}(G)$ is a real symmetric matrix, all its eigenvalues are real and can be listed as $\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_n$. The multiset of such eigenvalues together with their multiplicities is called the *distance spectrum* of G, denoted by $\mathrm{Sp}_D(G) = \{[\lambda_1]^{m_1}, \dots, [\lambda_s]^{m_s}\}$ where $\lambda_1, \dots, \lambda_s$ are all distinct eigenvalues and m_i is the multiplicity of λ_i. For more details about the distance matrix we refer the readers to [2].

As usual, we always write C_n and K_n for the cycle and the complete graph of corresponding orders. For two graphs G and H with disjoint vertex sets, the *union* $G \cup H$ is the graph with vertex set $V = V(G) \cup V(H)$ and edge set $E = E(G) \cup E(H)$. The union of k copies of G is denoted by kG. The *join* $G \nabla H$ is the graph obtained from $G \cup H$ by adding all edges between $V(G)$ and $V(H)$. The graph $W(n) = K_1 \nabla C_n$ is called the *wheel graph* and the graph $GW(a, m, n) = aK_m \nabla C_n$ is called the *generalized wheel graph*. Clearly, a wheel graph is just a special case of generalized wheel graphs, i.e., $W(n) = GW(1, 1, n)$.

A graph is called *integral* if all eigenvalues of its adjacency matrix are integers. The problem to characterize integral graphs dates back to 1973, when Harary and Schwenk [7] posed the question “Which graphs have integral spectra?”. This problem initiated a significant investigation among algebraic graph theorists, trying to construct and classify integral graphs. Although this problem is easy to state, it turns out to be extremely hard. It has been attacked by many mathematicians during the past 40 years [1,3,5,6,10,14], and it is still wide open. With respect to distance matrix, a connected graph G is called *distance integral* if all eigenvalues of its distance matrix are integers. Although there is a huge amount of papers that study distance spectrum of graphs and their applications to distance energy of graphs, there are few researches on distance integral graph. In 2010, Ilić [9] determined the distance eigenvalues of integral circulant graphs and proved that these graphs are also distance integral. In 2011, Renteln [12] characterized the integral Cayley graphs over the Coxeter group.

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https://doi.org/10.1016/j.disc.2022.113258

In 2015, Pokorný et al. [11] gave some conditions for the distance integrality on graphs similar to complete split graphs. Very recently, Huang [8] gave some necessary and sufficient conditions for the distance integrality of Cayley graphs over dihedral groups. However, most of these works only consider Cayley graphs, whose distance eigenvalues could be expressed by using irreducible characters of the corresponding groups. For general graphs, it is not easy to determine whether they are distance integral or not. Therefore, we would like to investigate some general distance integral graphs. Note that the distance eigenvalues of the join of two regular graphs G_1 and G_2 are determined by the adjacency eigenvalues of G_1 and G_2. It seems to be a good start to investigate the distance integrality of $G_1 \nabla G_2$. Up to now, the distance integrality of $G_1 \nabla G_2$ is solved only when G_1 and G_2 are the disjoint unions of complete graphs or empty graphs. To move a step on, we would like to study $G_1 \nabla G_2$ when G_1 is the union of complete graphs but G_2 is a general graph. As a start, we make a lot of examples and the graph $aK_m \nabla C_n$, the so-called generalized wheel graph, comes to our eyes. In this paper, we completely determine all integral generalized wheel graphs.

2. Distance spectrum of $GW(a, m, n)$

In this part, we would like to give the distance spectrum of $GW(a, m, n)$. Note that $GW(a, m, n) = aK_m \nabla C_n$ is the join of two regular graphs. The following result is usable.

Lemma 1 *([13]). For $i = 1, 2$, let G_i be an r_i-regular graph with n_i vertices. If the eigenvalues of the adjacency matrix of G_i are given by $\lambda_1^{(i)} = r_i \geq \cdots \geq \lambda_{n_i}^{(i)}$, then the distance spectrum of $G_1 \nabla G_2$ consists of the eigenvalues $-\lambda_j^{(i)} - 2$ for $i = 1, 2$ and $j = 2, 3, \cdots, n_i$, and two more simple eigenvalues*

$$n_1 + n_2 - 2 - \frac{r_1 + r_2}{2} \pm \sqrt{\left(n_1 - n_2 - \frac{r_1 - r_2}{2}\right)^2 + n_1 n_2}.$$

According to Lemma 1, we need the adjacency spectra of aK_m and C_n to obtain the distance spectrum of $GW(a, m, n)$.

Lemma 2 *([4]). The adjacency spectrum of aK_m is $\left\{[m-1]^a, [-1]^{a(n-1)}\right\}$ and the adjacency spectrum of C_n is $\{2\cos(2\pi j/n) \mid 1 \leq j \leq n-1\}$.*

Combining Lemmas 1 and 2, we get the distance spectrum of $GW(a, m, n)$ immediately.

Theorem 1. *For positive integers a, m and $n \geq 3$, the distance spectrum of $GW(a, m, n)$ is given by*

$$\{\lambda_1, \lambda_2, [-m-1]^{a-1}, [-1]^{a(m-1)}, -2\cos(2\pi j/n) - 2 \mid 1 \leq j \leq n-1\},$$

where $\lambda_{1,2} = \frac{(2a-1)m+2n-5}{2} \pm \frac{1}{2}\sqrt{((2a-1)m - 2n + 3)^2 + 4amn}$.

From Theorem 1, we get the distance spectrum of $GW(1, m, n)$

Corollary 1. *For positive integer m and $n \geq 3$, the distance spectrum of $GW(1, m, n)$ is given by*

$$\{\beta_1, \beta_2, [-1]^{m-1}, -2\cos(2\pi j/n) - 2 \mid 1 \leq j \leq n-1\},$$

where $\beta_{1,2} = \frac{m+2n-5}{2} \pm \frac{1}{2}\sqrt{(m-2n+3)^2 + 4mn}$.

3. Distance integral generalized wheel graphs

In this part, we completely determine all distance integral generalized wheel graphs. We first consider $GW(a, m, n)$ for the special case $a = 1$.

Lemma 3. *The generalized wheel graph $GW(1, m, n)$ is distance integral if and only if one of the following holds*
(i) $n = 3$ and $m \geq 1$;
(ii) $n = 6$ and $m = 4$;
(iii) $n = 6$ and $m = 14$.

Proof. From Corollary 1, the distance spectrum of $GW(1, m, n)$ consists of $[-1]^{m-1}$, $-2\cos(2\pi j/n) - 2$ and two simple eigenvalues

$$\frac{m+2n-5}{2} \pm \frac{1}{2}\sqrt{(m-2n+3)^2 + 4mn}, \tag{1}$$

where $1 \le j \le n-1$. Since $m+2n-5 \equiv m-2n+3 \pmod 2$, we have $m+2n-5 \equiv (m-2n+3)^2+4mn \pmod 2$. Therefore, the numbers in (1) are integers if and only if $(m-2n+3)^2+4mn$ is a perfect square. Thus, the graph $GW(1,m,n)$ is distance integral if and only if $-2\cos(2\pi j/n)$ is integral for any $1 \le j \le n-1$ and $(m-2n+3)^2+4mn$ is a perfect square. By immediate calculations, $-2\cos(2\pi j/n)$ is integral for any j and $(m-2n+3)^2+4mn$ is a perfect square when either (i), (ii) or (iii) holds. In what follows, we consider the other side.

It is clear that $-2\cos(2\pi j/n)$ is integral for any $1 \le j \le n-1$ if and only if $n \in \{3,4,6\}$. Denote by $t=(m-2n+3)^2+4mn$. If $n=3$ then $t=(m-3)^2+12m=(m+3)^2$. Therefore, the graph $K_m \nabla C_3$ is always distance integral for any m. If $n=4$ then $t=(m-5)^2+16m=m^2+6m+25=(m+3)^2+16$. Assume that $t=c^2$. We have $(m+3)^2+16=c^2$, which is equivalent to $16=(c+(m+3))(c-(m+3))$. It leads to $c=5$ and $m+3=3$, or $c=4$ and $m+3=0$, which are all impossible. If $n=6$ then $t=(m-9)^2+24m=(m+3)^2+72$. Assume that $t=c^2$. We have $(m+3)^2+72=c^2$, which indicates $72=(c+(m+3))(c-(m+3))$. It leads to either $c=19$ and $m+3=17$, $c=11$ and $m+3=7$, or $c=9, m+3=3$. The first case leads to $m=14$, the second one leads to $m=4$, and the last one is impossible.

The proof is completed. $\square$

In what follows, we turn our eyes on the generalized wheel graphs $GW(a,m,n)$ with $a \ge 2$.

Lemma 4. *For positive integers a, m, n with $a \ge 2$ and $n \ge 3$, the generalized wheel graph $GW(a,m,n)$ is distance integral if and only if the integers a, m, n satisfy one of the following conditions*
(i) $n=3$ and

$$m=\frac{\alpha^2-6\alpha-36a(a-1)}{2\alpha(2a-1)^2}$$

for some $\alpha \in \mathbb{N}$ with $\alpha > 6a$.
(ii) $n=4$ and

$$m=\frac{\alpha^2-2(5-2a)\alpha-16a(6a-5)}{2\alpha(2a-1)^2},$$

for some $\alpha \in \mathbb{N}$ with $\alpha \ge \max\{8a+1, \sqrt{16a(6a-5)}\}$, or

$$m=\frac{-\alpha^2-2(5-2a)\alpha+16a(6a-5)}{2\alpha(2a-1)^2},$$

for some $\alpha \in \mathbb{N}$ with $\sqrt{16a(6a-5)} \le \alpha < 12a-10$.
(iii) $n=6$ and

$$m=\frac{\alpha^2-2(9-6a)\alpha-72a(4a-3)}{2\alpha(2a-1)^2},$$

for some $\alpha \in \mathbb{N}$ with $\alpha \ge \max\{12a+1, \sqrt{72a(4a-3)}\}$, or

$$m=\frac{-\alpha^2-2(9-6a)\alpha+72a(4a-3)}{2\alpha(2a-1)^2},$$

for some $\alpha \in \mathbb{N}$ with $\sqrt{72a(4a-3)} \le \alpha < 24a-18$.

Proof. From Theorem 1, the distance spectrum of $GW(a,m,n)$ consists of $[-m-1]^{a-1}, [-1]^{a(m-1)}$, $-2\cos(2\pi j/n)-2$ and two simple eigenvalues

$$\frac{(2a-1)m+2n-5}{2} \pm \frac{1}{2}\sqrt{((2a-1)m-2n+3)^2+4amn}, \tag{2}$$

where $1 \le j \le n-1$. Since $(2a-1)m+2n-5 \equiv (2a-1)m-2n+3 \pmod 2$, we have $(2a-1)m+2n-5 \equiv ((2a-1)m-2n+3)^2+4amn \pmod 2$. Therefore, the numbers in (2) are integers if and only if $((2a-1)m-2n+3)^2+4amn$ is a perfect square. Thus, the graph $GW(a,m,n)$ is distance integral if and only if $2\cos(2\pi j/n)$ is integral for any $1 \le j \le n-1$ and $((2a-1)m-2n+3)^2+4amn$ is a perfect square. By immediate calculations, if either (i), (ii) or (iii) holds, then $2\cos(2\pi j/n)$ is integral for any j and $((2a-1)m-2n+3)^2+4amn$ is a perfect square. In what follows, we consider the other side.

It is clear that $-2\cos(2\pi j/n)$ is integral for any $1 \le j \le n-1$ if and only if $n \in \{3,4,6\}$. Denote by $t=((2a-1)m-2n+3)^2+4amn$. It needs to divide the following cases to discuss.

Case 1. $n=3$.

In this case, we have $t=(2a-1)^2m^2+6m+9>9$. Assume that $t=c^2$ for some positive integer $c>3$. We have $(2a-1)^2m^2+6m+9=c^2$, and thus $m=\frac{-3\pm\sqrt{(2a-1)^2c^2-36a(a-1)}}{(2a-1)^2}$. Since m is an integer, we have $(2a-1)^2c^2-36a(a-1)=$

p^2 is a perfect square for some $p \geq 0$. It leads to $m = \frac{-3\pm p}{(2a-1)^2}$ and $36a(a-1) = (2a-1)^2c^2 - p^2$. Therefore, $\frac{36a(a-1)}{(2a-1)c-p} = (2a-1)c + p = \alpha$ is a positive integer. It follows that

$$\begin{cases} \alpha = (2a-1)c + p, \\ 36a(a-1) = (2a-1)\alpha c - \alpha p. \end{cases} \tag{3}$$

Adding α times of the first equation in (3) to the second one, we get $\alpha^2 + 36a(a-1) = 2(2a-1)\alpha c$. It means $c = \frac{\alpha^2+36a(a-1)}{2(2a-1)\alpha}$, and thus $p = \frac{\alpha^2-36a(a-1)}{2\alpha}$. Therefore, we have $m = \frac{-3+p}{(2a-1)^2} = \frac{\alpha^2-6\alpha-36a(a-1)}{2\alpha(2a-1)^2}$ or $m = \frac{-3-p}{(2a-1)^2} = \frac{-\alpha^2-6\alpha+36a(a-1)}{2\alpha(2a-1)^2}$. Note that $\alpha \geq \sqrt{36a(a-1)}$ due to $p \geq 0$. Moreover, if $m = \frac{\alpha^2-6\alpha-36a(a-1)}{2\alpha(2a-1)^2}$ then we get $\alpha > 6a$ according to $m > 0$; if $m = \frac{-3-p}{(2a-1)^2} = \frac{-\alpha^2-6\alpha+36a(a-1)}{2\alpha(2a-1)^2}$ then we have $\alpha < 6a-6$ according to $m > 0$ which contradicts $\alpha \geq \sqrt{36a(a-1)} > 6a-6$. Thus, (i) holds.

Case 2. $n = 4$.

In this case, we have $t = (2a-1)^2m^2 + (10-4a)m + 25$. Assume that $t = c^2$ for some integer $c \geq 0$. We have $(2a-1)^2m^2 + (10-4a)m + 25 = c^2$, and thus $m = \frac{(2a-5)\pm\sqrt{(2a-1)^2c^2-16a(6a-5)}}{(2a-1)^2}$. Since m is an integer, we have $(2a-1)^2c^2 - 16a(6a-5) = p^2$ is a perfect square for some $p \geq 0$. It leads to $m = \frac{(2a-5)\pm p}{(2a-1)^2}$ and $16a(6a-5) = (2a-1)^2c^2 - p^2$. Therefore, $\frac{16a(6a-5)}{(2a-1)c-p} = (2a-1)c + p = \alpha$ is a positive integer. It follows that

$$\begin{cases} \alpha = (2a-1)c + p, \\ 16a(6a-5) = (2a-1)\alpha c - \alpha p. \end{cases} \tag{4}$$

Adding α times of the first equation in (4) to the second one, we get $\alpha^2 + 16a(6a-5) = 2(2a-1)\alpha c$. It means $c = \frac{\alpha^2+16a(6a-5)}{2(2a-1)\alpha}$, and thus $p = \frac{\alpha^2-16a(6a-5)}{2\alpha}$. Therefore, we have $m = \frac{(2a-5)+p}{(2a-1)^2} = \frac{\alpha^2+2(2a-5)\alpha-16a(6a-5)}{2\alpha(2a-1)^2}$ or $m = \frac{(2a-5)-p}{(2a-1)^2} = \frac{-\alpha^2+2(2a-5)\alpha+16a(6a-5)}{2\alpha(2a-1)^2}$. Note that $\alpha \geq \sqrt{16a(6a-5)}$ due to $p \geq 0$. Moreover, if $m = \frac{\alpha^2+2(2a-5)\alpha-16a(6a-5)}{2\alpha(2a-1)^2}$ then we get $\alpha > 8a$ according to $m > 0$; if $m = \frac{(2a-5)-p}{(2a-1)^2} = \frac{-\alpha^2+2(2a-5)\alpha+16a(6a-5)}{2\alpha(2a-1)^2}$ then we have $\alpha < 12a-10$ according to $m > 0$. Thus, (ii) holds.

Case 3. $n = 6$.

In this case, we have $t = (2a-1)^2m^2 + (18-12a) + 81$. Assume that $t = c^2$ for some integer $c \geq 0$. We have $(2a-1)^2m^2 + (18-12a)m + 81 = c^2$, and thus $m = \frac{(6a-9)\pm\sqrt{(2a-1)^2c^2-72a(4a-3)}}{(2a-1)^2}$. Since m is an integer, we have $(2a-1)^2c^2 - 72a(4a-3) = p^2$ is a perfect square for some $p \geq 0$. It leads to $m = \frac{(6a-9)\pm p}{(2a-1)^2}$ and $72a(4a-3) = (2a-1)^2c^2 - p^2$. Therefore, $\frac{72a(4a-3)}{(2a-1)c-p} = (2a-1)c + p = \alpha$ is a positive integer. It follows that

$$\begin{cases} \alpha = (2a-1)c + p, \\ 72a(4a-3) = (2a-1)\alpha c - \alpha p. \end{cases} \tag{5}$$

Adding α times of the first equation in (5) to the second one, we get $\alpha^2 + 72a(4a-3) = 2(2a-1)\alpha c$. It means $c = \frac{\alpha^2+72a(4a-3)}{2(2a-1)\alpha}$, and thus $p = \frac{\alpha^2-72a(4a-3)}{2\alpha}$. Therefore, we have $m = \frac{(6a-9)+p}{(2a-1)^2} = \frac{\alpha^2+2(6a-9)\alpha-72a(4a-3)}{2\alpha(2a-1)^2}$ or $m = \frac{(6a-9)-p}{(2a-1)^2} = \frac{-\alpha^2+2(6a-9)\alpha+72a(4a-3)}{2\alpha(2a-1)^2}$. Note that $\alpha \geq \sqrt{72a(4a-3)}$ due to $p \geq 0$. Moreover, if $m = \frac{\alpha^2+2(6a-9)\alpha-72a(4a-3)}{2\alpha(2a-1)^2}$ then we get $\alpha > 12a$ according to $m > 0$; if $m = \frac{(6a-9)-p}{(2a-1)^2} = \frac{-\alpha^2+2(6a-9)\alpha+72a(4a-3)}{2\alpha(2a-1)^2}$ then we have $\alpha < 24a-18$ according to $m > 0$. Thus, (iii) holds. □

Though Lemma 4 gives the necessary and sufficient condition for $GW(a, m, n)$ to be distance integral when $a \geq 2$, it is hard to determine which numbers a, m, n could satisfy the conditions (i) (ii) or (iii). We even do not know whether there exist such integers or not. Therefore, we need to do further research to investigate the distance integrality of $GW(a, m, n)$.

Corollary 2. *Let $a \geq 2$, $m \geq 1$ and $n \geq 3$ be positive integers such that the graph $GW(a, m, n)$ is distance integral. It holds that*
(i) if $n = 3$ then $m \leq 2$;
(ii) if $n = 4$ then $m \leq 6$;
(iii) if $n = 6$ then $m \leq 20$.

Proof. If $n = 3$, Lemma 4 indicates $m = \frac{\alpha^2-6\alpha-36a(a-1)}{2\alpha(2a-1)^2}$ for some $\alpha \in \mathbb{N}$ with $\alpha > 6a$. Since m is an integer, we have $2\alpha(2a-1)^2 \mid \alpha^2 - 6\alpha - 36a(a-1)$. If α is even, then $2\alpha \mid \alpha^2 - 6\alpha$ and thus $2\alpha \mid 36a(a-1)$. Therefore, we have $\alpha \mid 18a(a-1)$. If α is odd then $gcd(\alpha, 4) = 1$ and thus $\alpha \mid 9a(a-1)$ due to $\alpha \mid 36a(a-1)$. Hence we always have $\alpha \mid 18a(a-1)$ and thus $\alpha \leq 18a(a-1)$ because $18a(a-1) > 0$.

Suppose to the contrary that $m \geq 3$. We have $\alpha^2 - 6\alpha - 36a^2 + 36a \geq 6(2a-1)^2\alpha$. It yields $\alpha \geq 12a^2 - 12a + 6 + (12a-6)\sqrt{a^2-a+1}$ or $\alpha \leq 12a^2 - 12a + 6 - (12a-6)\sqrt{a^2-a+1}$. If the former occurs, we have

$$\begin{aligned}\alpha &\geq 12a^2 - 12a + 6 + (12a-6)\sqrt{a^2-a+1}\\ &\geq 12a^2 - 12a + 6 + (12a-6)(a-1)\\ &= 24a^2 - 30a + 12 > 18a(a-1),\end{aligned}$$

which contradicts $\alpha \leq 18a(a-1)$. If the latter occurs, we have

$$\begin{aligned}\alpha &\leq 12a^2 - 12a + 6 - (12a-6)\sqrt{a^2-a+1}\\ &\leq 12a^2 - 12a + 6 - (12a-6)(a-1) = 6a,\end{aligned}$$

which contradicts $\alpha > 6a$. Thus, (i) holds.

If $n = 4$, it should be divided to two cases to discuss according to Lemma 4.

Case 1. $m = \frac{\alpha^2 - 2(5-2a)\alpha - 16a(6a-5)}{2\alpha(2a-1)^2}$ for some $\alpha \in \mathbb{N}$ with $\alpha > 8a$ and $\alpha \geq \sqrt{16a(6a-5)}$.

Since m is an integer, we have $2\alpha(2a-1)^2 \mid \alpha^2 - 2(5-2a)\alpha - 16a(6a-5)$. If α is even then $2\alpha \mid \alpha^2 - 2(5-2a)\alpha$. It means $2\alpha \mid 16a(6a-5)$. It leads to $\alpha \mid 8a(6a-5)$. If α is odd then $gcd(\alpha, 16) = 1$ and thus $\alpha \mid a(6a-5)$ due to $\alpha \mid 16a(6a-5)$. Hence we always have $\alpha \mid 8a(6a-5)$ and thus $\alpha \leq 8a(6a-5)$ because $8a(6a-5) > 0$.

Suppose to the contrary that $m \geq 7$. We have $\alpha^2 - 2(5-2a)\alpha - 96a^2 + 80a \geq 14(2a-1)^2\alpha$. It yields $\alpha \geq 28a^2 - 30a + 12 + (4a-2)\sqrt{49a^2 - 56a + 36}$ or $\alpha \leq 28a^2 - 30a + 12 - (4a-2)\sqrt{49a^2-56a+36}$. If the former occurs, we have

$$\begin{aligned}\alpha &\geq 28a^2 - 30a + 12 + (4a-2)\sqrt{49a^2-56a+36}\\ &> 28a^2 - 30a + 12 + (4a-2)(7a-4)\\ &= 56a^2 - 60a + 20 > 8a(6a-5),\end{aligned}$$

which contradicts $\alpha \leq 8a(6a-5)$. If the latter occurs, we have

$$\begin{aligned}\alpha &\leq 28a^2 - 30a + 12 - (4a-2)\sqrt{49a^2-56a+36}\\ &< 28a^2 - 30a + 12 - (4a-2)(7a-4) = 4,\end{aligned}$$

which contradicts $\alpha \geq \sqrt{16a(6a-5)}$.

Case 2. $m = \frac{-\alpha^2 - 2(5-2a)\alpha + 16a(6a-5)}{2\alpha(2a-1)^2}$ for some $\alpha \in \mathbb{N}$ with $\sqrt{16a(6a-5)} \leq \alpha < 12a - 10$.

Suppose to the contrary $m \geq 7$. We have $-\alpha^2 - 2(5-2a)\alpha + 96a^2 - 80a \geq 14(2a-1)^2\alpha$. It yields $-28a^2 + 30a - 12 - (4a-2)\sqrt{49a^2-56a+36} \leq \alpha \leq -28a^2 + 30a - 12 + (4a-2)\sqrt{49a^2-56a+36}$. We have

$$\begin{aligned}\alpha &\leq -28a^2 + 30a - 12 + (4a-2)\sqrt{49a^2-56a+36}\\ &< -28a^2 + 30a - 12 + (4a-2)(7a-3) = 4a - 6\\ &< \sqrt{16a(6a-5)},\end{aligned}$$

which contradicts $\alpha \geq \sqrt{16a(6a-5)}$.

Thus, (ii) holds.

If $n = 6$, it also should be divided to two cases to discuss according to Lemma 4.

Case 1. $m = \frac{\alpha^2 - 2(9-6a)\alpha - 72a(4a-3)}{2\alpha(2a-1)^2}$ for some $\alpha \in \mathbb{N}$ with $\alpha \geq \max\{12a+1, \sqrt{72a(4a-3)}\}$.

Since m is an integer, we have $2\alpha(2a-1)^2 \mid \alpha^2 - 2(9-6a)\alpha - 72a(4a-3)$. If α is even, then $2\alpha \mid \alpha^2 - 2(9-6a)\alpha$. It means $2\alpha \mid 72a(4a-3)$. It leads to $\alpha \mid 36a(4a-3)$. If α is odd, then $gcd(\alpha, 8) = 1$ and thus $\alpha \mid 9a(4a-3)$ due to $\alpha \mid 72a(4a-3)$. Hence we always have $\alpha \mid 36a(4a-3)$ and thus $\alpha \leq 36a(4a-3)$ because $36a(4a-3) > 0$.

Suppose to the contrary that $m \geq 21$. We have $\alpha^2 - 2(9-6a)\alpha - 288a^2 + 216a \geq 42(2a-1)^2\alpha$. It yields $\alpha \geq 84a^2 - 90a + 30 + (12a-6)\sqrt{49a^2-56a+25}$ or $\alpha \leq 84a^2 - 90a + 30 - (12a-6)\sqrt{49a^2-56a+25}$. If the former occurs, we have

$$\begin{aligned}\alpha &\geq 84a^2 - 90a + 30 + (12a-6)\sqrt{49a^2-56a+25}\\ &> 84a^2 - 90a + 30 + (12a-6)(7a-4)\\ &= 168a^2 - 180a + 54 > 36a(4a-3),\end{aligned}$$

which contradicts $\alpha \leq 36a(4a-3)$. If the latter occurs, we have

$$\begin{aligned}\alpha &\leq 84a^2 - 90a + 30 - (12a-6)\sqrt{49a^2-56a+25}\\ &< 84a^2 - 90a + 30 - (12a-6)(7a-4)\\ &= 6 < 12a,\end{aligned}$$

which contradicts $\alpha > 12a$.

Case 2. $m = \frac{-\alpha^2 - 2(9-6a)\alpha + 72a(4a-3)}{2\alpha(2a-1)^2}$ for $\alpha \in \mathbb{N}$, $\sqrt{72a(4a-3)} \leq \alpha < 24a - 18$.

Table 1
The case for $n = 3$.

m	t	corresponding equations	suitable solution
1	$(2a-1)^2+15$	$(c+(2a-1))(c-(2a-1))=15$	$c=8, a=4$
2	$(4a-2)^2+21$	$(c+(4a-2))(c-(4a-2))=21$	$c=11, a=3$

Table 2
The case for $n = 4$.

m	t	corresponding equations	suitable solution
1	$(2a-2)^2+32$	$(c+(2a-2))(c-(2a-2))=32$	$c=6, a=2$
2	$(4a-3)^2+40$	$(c+(4a-3))(c-(4a-3))=40$	$c=11, a=3$
6	$(12a-7)^2+72$	$(c+(12a-7))(c-(12a-7))=72$	$c=19, a=2$

Table 3
The case for $n = 6$.

m	t	corresponding equations	suitable solution
1	$(2a-4)^2+84$	$(c+(2a-4))(c-(2a-4))=84$	$\begin{cases} c=22, a=12 \\ c=10, a=4 \end{cases}$
2	$(4a-5)^2+96$	$(c+(4a-5))(c-(4a-5))=96$	$c=25, a=7$
3	$(6a-6)^2+108$	$(c+(6a-6))(c-(6a-6))=108$	$c=12, a=2$
5	$(10a-8)^2+132$	$(c+(10a-8))(c-(10a-8))=132$	$c=34, a=4$
10	$(20a-13)^2+192$	$(c+(20a-13))(c-(20a-13))=192$	$c=49, a=3$

Suppose to the contrary $m \geq 21$. We have $-\alpha^2 - 2(9-6a)\alpha + 288a^2 - 216a \geq 42(2a-1)^2\alpha$. It yields $-84a^2 + 90a - 30 - (12a-6)\sqrt{49a^2-56a+25} \leq \alpha \leq -84a^2 + 90a - 30 + (12a-6)\sqrt{49a^2-56a+25}$. We have

$$\begin{aligned} \alpha &\leq -84a^2 + 90a - 30 + (12a-6)\sqrt{49a^2-56a+25} \\ &< -84a^2 + 90a - 30 + (12a-6)(7a-3) \\ &= 12a - 12 < \sqrt{72a(4a-3)}, \end{aligned}$$

which contradicts $\alpha \geq \sqrt{72a(4a-3)}$.

Thus, (iii) holds. □

Combining Lemma 4 and Corollary 2, we get the following result.

Lemma 5. *For positive integers $a \geq 2$, $m \geq 1$ and $n \geq 3$, the graph $GW(a,m,n)$ is distance integral if and only if the ordered triple $[a,m,n] \in S$, where*

$$S = \{[4,1,3],[3,2,3],[2,1,4],[3,2,4],[2,6,4],[4,1,6],[12,1,6],[7,2,6],[2,3,6],[4,5,6],[3,10,6]\}.$$

Proof. According to Theorem 1, one can easily verify that $GW(a,m,n)$ is distance integral when $[a,m,n] \in S$ by immediate calculations. In what follows, we show the necessity.

Assume $GW(a,m,n)$ is distance integral. We have $t = [(2a-1)m - 2n + 3]^2 + 4amn = c^2$ is a perfect square according to Theorem 1. Moreover, Lemma 4 implies that $n \in \{3,4,6\}$.

If $n = 3$ then $(2a-1)m^2 + 6m + 9 = c^2$. Note that Corollary 2 indicates $m \leq 2$. If $m = 1$, then $c^2 = (2a-1)^2 + 15$, that is $15 = (c+(2a-1))(c-(2a-1))$. It leads to $c + (2a-1) = 15$ and $c - (2a-1) = 1$, or $c + (2a-1) = 5$ and $c - (2a-1) = 3$. The former case yields $c = 8$ and $a = 4$, and the latter case yields $c = 4$ and $a = 1$ which contradicts $a \geq 2$. Hence we have $c = 8$ and $a = 4$. If $m = 2$, then $c^2 = (4a-2)^2 + 21$, that is $21 = (c+(4a-2))(c-(4a-2))$. It leads to $c + (4a-2)) = 21$ and $c - (4a-2) = 1$, or $c + (4a-2) = 7$ and $c - (4a-2) = 3$. The former case indicates $c = 11$ and $a = 3$, and the latter case indicates $c = 5$ and $a = 1$, which contradicts $a \geq 2$. Hence we have $c = 11$ and $a = 3$. The critical parts of the proof could be presented in Table 1. Similarly, we can deal with the cases of $n = 4$ and $n = 6$. To avoid tedious repetitive processes, we omit the details and only present the key steps in Tables 2 and 3. □

Combining Lemmas 3 and 5, the main result is obtained.

Theorem 2. *The generalized wheel graph $GW(a,m,n)$ is distance integral if and only if one of the followings holds*
(i) $a = 1$, $n = 3$ and $m \geq 1$;
(2) the ordered triple $[a,m,n] \in \mathcal{S}$ where

$$\mathcal{S} = \{[1,4,6],[1,14,6],[4,1,3],[3,2,3],[2,1,4],[3,2,4],[2,6,4],[4,1,6],[12,1,6],[7,2,6],[2,3,6],[4,5,6],[3,10,6]\}.$$

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

This work is supported by NSFC (Nos. 12001544, 11671402, 11871479). The authors are so grateful to the referees for their valuable comments and corrections which improve the presentation of the paper.

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