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Signed Zero-Divisor Graphs Over Commutative Rings

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Abstract
In this paper, we define signed zero-divisor graphs over commutative rings and investigate the interplay between the algebraic properties of the rings and the combinatorial properties of their corresponding signed zero-divisor graphs. We investigate the structure of signed zero-divisor graphs, the relation between ideals and signed zero-divisor graphs, and the adjacency matrices and the spectra of signed zero-divisor graphs.

Keywords Ring · Ideal · Signed graph · Adjacency matrix

Mathematics Subject Classifications: 05C50 · 05E99 · 15B57

1 Introduction

The zero-divisor graphs are firstly defined over commutative finite rings to study the interplay between graph-theoretical properties of such graphs and ring-theoretical properties of the corresponding rings. The definition of zero-divisor graphs was initiated by Beck [4], who defined the zero-divisor graph $\Gamma(R)$ of a commutative ring R to be the graph with vertex set R and two vertices x and y are adjacent if and only if $xy = 0$. This concept was modified by Anderson and Livingston [2], who defined the zero-divisor graph of a commutative ring R to be the graph with vertex set $Z(R)^* = Z(R) \setminus \{0\}$, which is the set of non-zero zero-divisors, and two vertices x and y are adjacent if and only if $xy = 0$. This definition has been favored by many mathematicians and many significant contributions are obtained [1, 3, 13, 14].

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For a commutative ring R and a subset $S \subseteq R$, denote by $S^* = S \setminus \{0\}$ and $\overline{S} = R \setminus S$. Denote by $R_1 \oplus R_2$ the direct sum of two rings R_1 and R_2. An element $r \in R$ is nilpotent if there exists a positive integer k such that $r^k = 0$. Denote by $\mathrm{Nil}(R)$ the set of nilpotent elements of R. Clearly, $\mathrm{Nil}(R) \subseteq Z(R)$. Note that the classical zero-divisor graphs do not reveal the particularity of nilpotent elements. We would like to define a new graph, the signed zero-divisor graph, over a commutative ring which associates not only with the zero-divisors but also with the nilpotent elements of the ring. Before giving this definition, it needs to give another definition of zero-divisor graphs, which is a little different from that given by Anderson and Livingston.

Definition 1.1 Let R be a finite commutative ring. The zero-divisor graph $\Gamma(R)$ is the graph with vertex set $Z(R)^*$ and edge set $\{\{u, v\} \mid u, v \in Z(R)^*, uv = 0\}$, where u and v may be the same.

In this definition, the zero-divisor graphs may contain loops. For a graph Γ, which may contain loops, the reduced graph $\tilde{\Gamma}$ of Γ is the simple graph obtained from Γ by deleting all loops. Note that the reduced zero-divisor graph $\tilde{\Gamma}(R)$ is just the zero-divisor graph defined by Anderson and Livingston [2].

Next, we recall some definitions on signed graphs. A signed graph $\Phi = (\Gamma, \sigma)$ consists of a graph Γ and a sign function $\sigma\colon E(\Gamma) \to \{\pm 1\}$ on the edge set $E(\Gamma)$ of Γ. The graph Γ is the underlying graph of Φ. If two vertices u and v are adjacent in Γ, then we denote by $u \sim v$. An edge e is positive if $\sigma(e) = 1$, and negative if $\sigma(e) = -1$. The sign of a walk $P = u_1u_2\cdots u_k$ is $\sigma(P) = \prod_{i=1}^{k-1} \sigma(u_iu_{i+1})$. A walk P is positive if $\sigma(P) = 1$, and negative if $\sigma(P) = -1$. The signed graph is balanced if all its cycles are positive. The subgraph of Φ obtained from Φ by deleting all positive edges and then deleting all isolate vertices is the negative subgraph of Φ, denoted by Φ^-. The positive subgraph Φ^+ could be defined similarly. If $\Phi = \Phi^+$, then Φ is completely positive (or CP for short). If $\Phi = \Phi^-$, then Γ is completely negative (or CN for short). There are many insightful works on signed graphs, see, for example, [5, 10–12, 15]. For other notations in graph theory, one may refer to [7]. The adjacency matrix $A(\Phi)$ of a signed graph $\Phi = (\Gamma, \sigma)$ is the square matrix indexed by the vertices of Γ, and its (u, v)-entry is $\sigma(uv)$ if $u \sim v$ and 0 otherwise. The nullity and characteristic polynomial of Φ are, respectively, the nullity and characteristic polynomial of $A(\Phi)$. Since $A(\Phi)$ is a symmetric real matrix, all its eigenvalues are real and could be listed as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. These eigenvalues are called eigenvalues of Φ. The collection of all eigenvalues together with their multiplicities is the spectrum of Φ, denoted by $\mathrm{Sp}(\Phi) = \{[\lambda_1]^{m_1}, \ldots, [\lambda_s]^{m_s}\}$, where $\lambda_1, \ldots, \lambda_s$ are all distinct eigenvalues of Φ and m_i is the multiplicity of λ_i for $1 \leq i \leq s$. For more details on spectra of graphs, one may refer to [8].

Now we could give the definition of signed zero-divisor graphs.

Definition 1.2 Let R be a finite commutative ring. The signed zero-divisor graph $\Phi(R) = (\Gamma(R), \sigma)$ is the signed graph with underlying graph $\Gamma(R)$ and sign function $\sigma : E(\Gamma(R)) \to \{\pm 1\}$ defined by $\sigma(uv) = -1$ if $u, v \in \mathrm{Nil}(R)$ and 1 otherwise.

Remark 1.3 Since $\Gamma(R)$ may contain loops, the signed graph $\Phi(R)$ may also contain loops. The reduced signed zero-divisor graph $\tilde{\Phi}(R)$ is the signed graph obtained from

$\Phi(R)$ by deleting all loops. The underlying graph of $\tilde{\Phi}(R)$ is just $\tilde{\Gamma}(R)$. The negative subgraph $\Phi^-(R)$ is just the subgraph of $\Phi(R)$ induced by $\mathrm{Nil}(R)^*$, but $\Phi^+(R)$ may be not an induced subgraph of $\Phi(R)$.

This paper is organized as follows. In Sect. 2, we investigate some basic properties of signed zero-divisor graphs. In Sect. 3, we discuss the relation between ideals and signed zero-divisor graphs. In Sect. 4 we study the adjacency matrices and spectra of signed zero-divisor graphs. In Sect. 5, we propose some problems for further research.

2 Basic Properties

In this part, we present some basic properties of signed zero-divisor graphs. Since the edges of a signed graph Φ consist of the edges of Φ^+ and Φ^-, one could get the structure of Φ by investigating Φ^+ and Φ^-. By simple observations, we get the following result.

Lemma 2.1 *Let R be a finite commutative ring with signed zero-divisor graph $\Phi(R) = (\Gamma(R), \sigma)$. Then the following statements hold.*

(i) *$\Phi(R)$ is CN if and only if $Z(R) = Nil(R)$.*
(ii) *Each loop in $\Phi(R)$ is negative.*
(iii) *If $\Phi(R)$ has only one vertex, then there is a negative loop associated with it.*
(iv) *For $r \in R$, $r \in Nil(R)^*$ if and only if there is a negative edge incident to it.*
(v) *$\Phi(R)$ is CP if and only if it contains no loops.*

Proof Note that $\Phi^-(R)$ is the induced subgraph of $\Phi(R)$ with vertex set $\mathrm{Nil}(R)^*$, this leads to statement (i).

If there is a loop associated with r, then $r^2 = 0$. Therefore, $r \in \mathrm{Nil}(R)$ and thus this loop is negative. It yields (ii).

If $V(\Phi(R)) = \{r\}$, then $Z(R)^* = \{r\}$. Therefore, $r^2 = 0$ and thereby there is a negative loop incident to r. Thus (iii) holds. If $r \in \mathrm{Nil}(R)^*$, then there exists a positive integer n such that $r^n = 0$ and $r^{n-1} \neq 0$. Therefore, $r, r^{n-1} \in Z(R)^*$, and there is a negative edge between r and r^{n-1}. Conversely, if there is a negative edge incident to r, then $r \in \mathrm{Nil}(R)^*$ by the definition. Hence (iv) holds.

If $\Phi(R)$ is CP, then (ii) implies that it contains no loops. Conversely, assume that $\Phi(R)$ contains no loops. Suppose to the contrary that $\Phi(R)$ is not CP, that is, there is a negative edge incident to some vertex r. Therefore, $r^n = 0$ and $r^{n-1} \neq 0$ for some positive integer n. It leads to that $(r^{n-1})^2 = 0$, and thus there is a loop incident to r^{n-1}, a contradiction. Thus (v) holds. □

Remark 2.2 Lemma 2.1 (v) implies that if $\Phi(R)$ is CP, then $\Phi(R) = \tilde{\Phi}(R) = \Gamma(R) = \tilde{\Gamma}(R)$.

Recall that in a finite commutative ring R, each element of R is either a unit or a zero-divisor, and every non-unit of R is nilpotent if and only if R is a local ring.

Corollary 2.3 *Let R be a finite commutative ring. Then $\Phi(R)$ is CN if and only if R is local.*

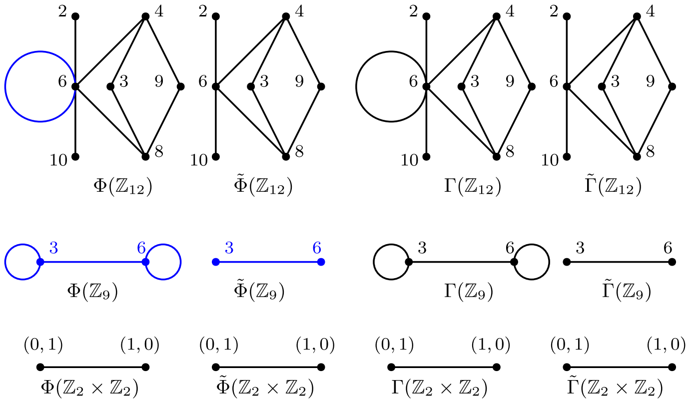

Fig. 1 The graphs in Example 2.4

To make the concepts more clear, we give some simple examples of signed zero-divisor graphs.

Example 2.4 In Fig. 1, we present the signed zero-divisor graphs, reduced signed zero-divisor graphs together with their underlying graphs over the rings $\mathbb{Z}_{12}$, $\mathbb{Z}_9$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, where the negative edges are colored blue and the positive edges are colored black. Note that, if $\tilde{\Gamma}(R_1) \neq \tilde{\Gamma}(R_2)$, then $\Phi(R_1) \neq \Phi(R_2)$. However, there exist R_1 and R_2 such that $\tilde{\Gamma}(R_1) = \tilde{\Gamma}(R_2)$ but $\Phi(R_1) \neq \Phi(R_2)$. Clearly, the rings $R_1 = \mathbb{Z}_9$ and $R_2 = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ are such rings. Therefore, the signed zero-divisor graphs could distinguish rings better than the classical zero-divisor graphs, that is, there are non-isomorphic rings share the same reduced zero-divisor graph but distinct signed zero-divisor graphs.

To study the properties of $\Phi(R)$, we consider the special case for $R = \mathbb{Z}_n$ firstly.

Example 2.5 By simple observations, the following statements are obtained.

- For a prime p and a positive integer m, the signed graph $\Phi(\mathbb{Z}_{p^m})$ is CN since $\mathbb{Z}_{p^m}$ is local.
- For distinct primes $p_1, p_2, \ldots, p_s$, the signed graph $\Phi(\mathbb{Z}_{p_1 p_2 \cdots p_s})$ is CP. Since otherwise, Lemma 2.1(vi) indicates that there is a loop incident to some vertex $r \in Z(R)^*$. It yields that $p_1 p_2 \cdots p_s \mid r^2$ and hence $r = 0$, a contradiction.
- For two distinct primes p and q, the signed graph $\tilde{\Phi}(\mathbb{Z}_{p^2 q})$ is CP and $\Phi(\mathbb{Z}_{p^2 q})$ is obtained from $\tilde{\Gamma}(\mathbb{Z}_{p^2 q})$ by adding a negative loop to the vertex pq. This follows from the fact that $\mathrm{Nil}(\mathbb{Z}_{p^2 q})^* = \{pq\}$.

For a graph Γ and two vertices $u, v \in V(\Gamma)$, the distance $d(u, v)$ between u and v is the length of a shortest path between them. The diameter $\dim(\Gamma)$ is the largest distance among all pairs of vertices in Γ, that is, $\mathrm{diam}(\Gamma) = \max\{d(u, v) \mid u, v \in V(\Gamma)\}$. The girth $\mathrm{gir}(\Gamma)$ of Γ is the length of a smallest cycle in Γ. It is proved in [2] that $\tilde{\Gamma}(R)$ is connected with $\mathrm{diam}(\tilde{\Gamma}(R)) \leq 3$ and if $\tilde{\Gamma}(R)$ contains a cycle, then $\mathrm{gir}(\tilde{\Gamma}(R)) \leq 7$. In case of $\Phi(R)$, we would like to investigate $\Phi^+(R)$ and $\Phi^-(R)$ to make the structure of $\Phi(R)$ more clear.

Theorem 2.6 *Let R be a finite commutative ring with identity. Then the following statements hold.*

(i) $\Phi^-(R)$ *is connected with diameter* $diam(\Phi^-(R)) \le 2$.
(ii) *If* $u, v \in V(\Phi^+(R)) \cap Nil(R)$, *then their distance* $d(u, v) \le 3$ *in* $\Phi^+(R)$.
(iii) *If* $u \in V(\Phi^+(R)) \cap Nil(R)$ *and* $v \in V(\Phi^+(R)) \setminus Nil(R)$, *then their distance* $d(u, v) \le 4$ *in* $\Phi^+(R)$. *Furthermore, if* $d(u, v) = 4$ *in* $\Phi^+(R)$, *then there exits* $x \in Z(R)^*$ *such that* $x^2 = 0$, $u \sim x$ *in* $\Phi^-(R)$ *and* $x \sim v$ *in* $\Phi^+(R)$.
(iv) *If* $u, v \in V(\Phi^+(R)) \setminus Nil(R)$, *then* $d(u, v) \le 5$ *in* $\Phi^+(R)$. *Furthermore, if* $d(u, v) > 3$, *then there exist* $u^*, v^* \in Z(R)^*$ *such that* $(u^*)^2 = (v^*)^2 = 0$, $u^* \sim v^*$ *in* $\Phi^-(R)$, *and* $u \sim u^*$, $v^* \sim v$ *in* $\Phi^+(R)$.

Proof Let $x, y \in \mathrm{Nil}(R)^*$ with $x^m = y^n = 0$. If $xy = 0$, then $x \sim y$ in $\Phi^-(R)$. Suppose that $xy \neq 0$. Define the set

$$X = \{(k_1, k_2) \mid x^{k_1} y^{k_2} \neq 0, 1 \le k_1 \le m-1, 1 \le k_2 \le n-1\}.$$

Clearly, $X \neq \emptyset$ as $xy \neq 0$. Taking $(a, b) \in X$ such that $a + b = \max\{k_1 + k_2 \mid (k_1, k_2) \in X\}$. Therefore, $x^{a+1} y^b = x^a y^{b+1} = 0$ and thereby $x \sim x^a y^b \sim y$ is a path of length 2 in $\Phi^-(R)$. Hence (i) holds.

Since R is finite commutative with identity, it can be expressed as the direct sum of local rings, that is, $R = R_1 \oplus \cdots \oplus R_n$ where R_i is local with identity for $1 \le i \le n$. Assume that ϵ_i is the identity of R_i for $1 \le i \le n$. Note that, for each R_i, there exists $w_i \in R_i^*$ such that $Z(R_i) = \mathrm{Nil}(R_i) = ann(w_i)$, where $ann(w_i) = \{r \in R_i \mid rw_i = 0\}$ is the annihilator of w_i. If R is local, then $V(\Phi^+(R)) = \emptyset$ and there is nothing to prove. Therefore, we may assume that R is not local, i.e., $n \ge 2$. For any $x = (x_1, \dots, x_n) \in R$, denote by $I_0(x) = \{1 \le i \le n \mid x_i = 0\}$ and $J(x) = \{1 \le j \le n \mid x_j \in Z(R_j)\}$. Clearly, $I_0(x) \subseteq J(x)$. For a subset $S \subseteq [n]$, let $\delta_S \in R$ be the element whose ith component is w_i if $i \in S$ and 0 otherwise, and let $e_S \in R$ be the element whose ith component is ϵ_i if $i \in S$ and 0 otherwise. If $S = \{a\}$, then we may write δ_a and e_a for δ_S and e_S, respectively. To prove the remaining statements, the following claims are needed.

Claim 2.7 *If* $u \in V(\Phi^+(R)) \cap Nil(R)$, *then* $I_0(u) \neq \emptyset$.

Proof If $I_0(u) = \emptyset$ and $v \sim u$ in $\Phi(R)$ for some $v = (v_1, \dots, v_n)$, then $uv = (u_1v_1, \dots, u_nv_n) = (0, \dots, 0)$. It means $u_iv_i = 0$ and thereby $v_i \in ann(w_i)$ for $1 \le i \le n$. Hence $v \in \mathrm{Nil}(R)$. It means that $\sigma(uv) = -1$ for any v adjacent to u, and thus $u \notin V(\Phi^+(R))$, a contradiction. □

Claim 2.8 *Let* $u, v \in Z(R)$ *be two elements with* $i_1 \in I_0(u)$ *and* $i_2 \in I_0(v)$. *If* $I_0(u) \cap I_0(v) = \emptyset$, *then* $u \sim e_{i_1} \sim e_{i_2} \sim v$ *in* $\Phi^+(R)$; *if there exits* $j \in I_0(u) \cap I_0(v)$, *then* $u \sim e_j \sim v$ *in* $\Phi^+(R)$.

Claim 2.9 *If* $u, v \in Z(R)^*$ *with* $i \in I_0(u) \cap J(v)$, *then* $u \sim \delta_i \sim v$ *in* $\Phi(R)$.

Claim 2.10 *If* $u \in Z(R) \setminus Nil(R)$ *and* $u' \sim u$ *in* $\Phi(R)$, *then* $\emptyset \neq J(u) \subset [n]$ *and* $\overline{J(u)} \subseteq I_0(u')$.

Proof Since $u \in Z(R)$, we have $J(u) \neq \emptyset$. Since $u \notin \mathrm{Nil}(R)$, $J(u)$ is a proper subset of $[n]$. Assume $u = (u_1, \ldots, u_n)$ and $u' = (u'_1, \ldots, u'_n)$. Since $u \sim u'$, we have $u_i u'_i = 0$ for $1 \leq i \leq n$. It leads to $u'_i = 0$ for $i \notin J(v)$ because u_i is a unit. Thus $\overline{J(u)} \subseteq I_0(u')$. □

Now we show (ii). For $u, v \in V(\Phi^+(R)) \cap \mathrm{Nil}(R)$, Claim 2.7 means $I_0(u), I_0(v) \neq \emptyset$. Therefore, Claim 2.8 implies the distance between them is at most 3 in $\Phi^+(R)$. Hence (ii) holds.

Next, we prove (iii). Assume that $u \in V(\Phi^+(R)) \cap \mathrm{Nil}(R)$ and $v \in V(\Phi^+(R)) \setminus \mathrm{Nil}(R)$. Let v' be a neighbor of v. Claim 2.10 indicates that $\overline{J(v)} \subseteq I_0(v')$ and thus $I_0(v') \neq \emptyset$. By Claim 2.8, the distance between u and v' is at most 3 in $\Phi^+(R)$, and thus $d(u, v) \leq 4$ in $\Phi^+(R)$. If $d(u, v) = 4$, then $d(u, v') = 3$ and thus $I_0(u) \cap I_0(v') = \emptyset$ due to Claim 2.8. Therefore, $I_0(u) \cap \overline{J(v)} = \emptyset$ and thereby $I_0(u) \cap J(v) \neq \emptyset$. By Claim 2.9, we have $u \sim \delta_i \sim v$ for some $i \in I_0(u) \cap J(v)$. If w_i is unit, then $\delta_i \notin \mathrm{Nil}(R)$ and $u \sim \delta_i \sim v$ forms a path of length 3 in $\Phi^+(R)$, which contradicts the assumption $d(u, v) = 4$ in $\Phi^+(R)$. Therefore, $w_i \in ann(w_i)$ and $u \sim \delta_i$ in $\Phi^-(R)$. Thus the second part of (iii) holds by taking $x = \delta_i$. Hence (iii) holds.

At last, we prove (iv). Assume $u, v \in V(\Phi^+(R)) \setminus \mathrm{Nil}(R)$ and $u' \sim u$ and $v' \sim v$ in $\Phi(R)$. Claim 2.10 indicates $\overline{J(u)} \subseteq I_0(u')$ and $\overline{J(v)} \subseteq I_0(v')$, and thereby $I_0(u'), I_0(v') \neq \emptyset$. Claim 2.8 implies $d(u', v') \leq 3$ and thus $d(u, v) \leq 5$ in $\Phi^+(R)$. If $d(u, v) > 3$, then $J(u) \cap J(v) = \emptyset$ since otherwise $u \sim \delta_j \sim v$ is a path in $\Phi^+(R)$ of length 2 for any $j \in J(u) \cap J(v)$. Therefore, $u \sim \delta_{J(u)} \sim \delta_{J(v)} \sim v$. If $\delta_{J(u)} \notin \mathrm{Nil}(R)$ or $\delta_{J(v)} \notin \mathrm{Nil}(R)$ then $u \sim \delta_{J(u)} \sim \delta_{J(v)} \sim v$ is a path in $\Phi^+(R)$ of length 3, contradicts the assumption $d(u, v) > 3$ in $\Phi^+(R)$. Thus, $\delta_{J(u)}, \delta_{J(v)} \in \mathrm{Nil}(R)$ and the second part of (iv) holds by taking $u^* = \delta_{J(u)}$ and $v^* = \delta_{J(v)}$. Hence (iv) holds.

The proof is completed. □

The following result is immediate from Theorem 2.6.

Corollary 2.11 *If R is a finite commutative ring with identity, then the following statements hold.*

- *$\Phi^+(R)$ is connected with diameter at most* 5.
- *If there exist $u \in Nil(R)^*$ and $v \in Z(R) \setminus Nil(R)$ such that $d(u, v) = 4$ in $\Phi^+(R)$, then the girth* $\mathrm{gir}(\tilde{\Gamma}(R)) \leq 6$.
- *If $d(u, v) = 5$ in $\Phi^+(R)$, then $u, v \in Z(R) \setminus Nil(R)$.*
- *If* $\mathrm{diam}(\Phi^+(R)) \geq 4$, *then* $\mathrm{gir}(\tilde{\Gamma}(R)) \leq 7$.

Example 2.12 In $\Phi^+(\mathbb{Z}_{36})$, the neighborhood of 2 is $\{18\}$ and the neighborhood of 3 is $\{12, 24\}$. Since 18, 12, 24 are all nilpotent, there are no edges between them in $\Phi^+(R)$. Therefore, $\mathrm{diam}(\Phi^+(\mathbb{Z}_{36})) > 3$ and thus $\mathrm{gir}(\tilde{\Gamma}(\mathbb{Z}_{36})) \leq 7$. In fact, $18 \sim 4 \sim 9 \sim 12 \sim 18$ is a cycle of length 4 in $\tilde{\Gamma}(\mathbb{Z}_{36})$.

Since $\Gamma(R)$ contains both the edges of $\Phi^+(R)$ and the edges of $\Phi^-(R)$, Theorem 2.6 implies the following result.

Corollary 2.13 ([2]) *If R is a finite commutative ring with identity, then its zero-divisor graph $\Gamma(R)$ is connected with diameter not greater than* 3.

Combining Corollary 2.3 and Theorem 2.6, we get the following result.

Corollary 2.14 *If R is a finite commutative local ring, then* $\mathrm{diam}(\Phi(R)) = \mathrm{diam}(\tilde{\Gamma}(R)) \leq 2$.

Recall that a signed graph is balanced if all its cycles are balanced. Since $\Phi(R)$ has a negative loop whenever it is not CP, the graph $\Phi(R)$ is balanced if and only if it is CP. A natural question is whether the $\tilde{\Phi}(R)$ is balanced when $\Phi(R)$ is not CP. For example, $\tilde{\Phi}(\mathbb{Z}_9)$ and $\tilde{\Phi}(\mathbb{Z}_{12})$ are balanced and $\tilde{\Phi}(\mathbb{Z}_{27})$ is not balanced because $3 \sim 9 \sim 18 \sim 3$ forms a negative cycle. Note that Example 2.5 gives that $\tilde{\Phi}(\mathbb{Z}_{p^2q})$ is CP and thus is balanced. In case of $\mathbb{Z}_n$, we have the following result.

Theorem 2.15 *Let n be a positive integer with prime factorization $n = 2^l p_1^{m_1} p_2^{m_2} \cdots p_s^{m_s}$, where $p_1, \ldots, p_s$ are distinct odd primes and $m_1 \geq m_2 \geq \cdots m_s \geq 0$. Then $\tilde{\Phi}(\mathbb{Z}_n)$ is balanced if and only if either $l = 2$ and $m_1 \leq 1$, or $l \leq 1$, $m_1 \leq 2$ and $m_2 \leq 1$.*

Proof If $l = 2$ and $m_1 \leq 1$, then $\mathrm{Nil}(\mathbb{Z}_n)^* = \{2p_1^{m_1} \cdots p_s^{m_s}\}$. It leads to that $\tilde{\Phi}(\mathbb{Z}_n)$ is CP and hence balanced. If $l \leq 1$ and $m_1 \leq 1$, then $\mathrm{Nil}(\mathbb{Z}_n)^* = \emptyset$ and thus $\Phi(R)$ is CP. It means $\tilde{\Phi}(R)$ is CP, and thereby is balanced. If $l \leq 1$ and $m_1 = 2$, then $\mathrm{Nil}(\mathbb{Z}_n)^* = \{2^l p_1 p_2^{m_2} \cdots p_s^{m_s}\}$. It leads to that $\tilde{\Phi}(\mathbb{Z}_n)$ is CP and hence balanced. The sufficiency holds, and we prove the necessity in what follows.

Assume that $\tilde{\Phi}(\mathbb{Z}_n)$ is balanced. Firstly, we show that $l \leq 3$ and $m_1 \leq 2$. If $l \geq 4$, then

$$2^{l-2} p_1^{m_1} \cdots p_s^{m_s} \sim 2^{l-1} p_1^{m_1} \cdots p_s^{m_s} \sim 3 \cdot 2^{l-2} p_1^{m_1} \cdots p_s^{m_s} \sim 2^{l-2} p_1^{m_1} \cdots p_s^{m_s}$$

forms a negative cycle, a contradiction. If $m_1 \geq 3$, then

$$2^l p_1^{m_1-2} p_2^{m_2} \cdots p_s^{m_s} \sim 2^l p_1^{m_1-1} p_2^{m_2} \cdots p_s^{m_s} \sim 2^{l+1} p_1^{m_1-2} p_2^{m_2} \cdots p_s^{m_s} \sim 2^l p_1^{m_1-2} p_2^{m_2} \cdots p_s^{m_s}$$

forms a negative cycle, a contradiction.

Secondly, we show that $m_2 \leq 1$. If $m_2 \geq 2$, then

$$2^l p_1^{m_1-1} p_2^{m_2-1} \cdots p_s^{m_s} \sim 2^l p_1^{m_1-1} p_2^{m_2} \cdots p_s^{m_s} \sim 2^l p_1^{m_1} p_2^{m_2-1} \cdots p_s^{m_s} \sim 2^l p_1^{m_1-1} p_2^{m_2-1} \cdots p_s^{m_s}$$

forms a negative cycle, a contradiction.

Thirdly, we show that $m_1 \leq 1$ whenever $l \geq 2$. If $l \geq 2$ and $m_1 \geq 2$, then

$$2^{l-1} p_1^{m_1-1} p_2^{m_2} \cdots p_s^{m_s} \sim 2^{l-1} p_1^{m_1} p_2^{m_2} \cdots p_s^{m_s} \sim 2^l p_1^{m_1-1} p_2^{m_2} \cdots p_s^{m_s} \sim 2^{l-1} p_1^{m_1-1} p_2^{m_2} \cdots p_s^{m_s}$$

forms a negative cycle, a contradiction.

Thus, it remains to show that the case $l = 3$ cannot happen. In fact, if $l = 3$, then it is proved $m_1 \leq 1$, and thereby $\mathrm{Nil}(\mathbb{Z}_n)^* = \{r_1, r_2\}$, where $r_1 = 2p_1^{m_1} \cdots p_s^{m_s}$ and $r_2 = 2^2 p_1^{m_1} \cdots p_s^{m_s}$. Moreover, $e = \{r_1, r_2\}$ is an edge in $\Phi^-(\mathbb{Z}_n)$. Note that $2^2 \sim r_1$ and $2 \sim r_2$ in $\Phi^+(\mathbb{Z}_n)$. Therefore, Theorem 2.6 (ii) implies that there is a path P of length at most 3 from r_1 to r_2. Thus, the path P and the edge e form a negative cycle, a contradiction.

The proof is completed. □

From the proof of Theorem 2.15, we get the following result.

Corollary 2.16 *The signed graph $\tilde{\Phi}(\mathbb{Z}_n)$ is balanced if and only if it is CP.*

3 Ideals and Signed Zero-Divisor Graphs

In this part, we discuss the relation between the ideal I of the ring R and the structure of $\Phi(R)$. For an ideal I of the ring R, the nil radical of I, denoted by $\sqrt{I}$, is the set

$$\sqrt{I} = \{r \in R \mid r^k \in I \text{ for some positive integer } k \ (k \text{ varies with } r)\}.$$

The ideal I is prime if $ab \in I$ indicates that $a \in I$ or $b \in I$, I is semiprime if $I = \sqrt{I}$, and is primary if the conditions $ab \in I$ and $a \notin I$ together imply $b \in \sqrt{I}$. We start with the following classical result.

Lemma 3.1 ([6]) *An ideal I of R is prime if and only if $Z(R/I) = \{\overline{0}\}$, is primary if and only if $Z(R/I) = Nil(R/I)$, and is semiprime if and only if $Nil(R/I) = \{\overline{0}\}$.*

According to the definitions, Lemma 3.1 could be rewritten as follows.

Corollary 3.2 *An ideal I of R is prime if and only if $\Phi(R/I)$ is a null graph, is primary if and only if $\Phi(R/I)$ is CN, and is semiprime if and only if $\Phi(R/I)$ is CP.*

Example 3.3 The signed zero-divisor graphs $\Phi(\mathbb{Z}_{12})$, $\Phi(\mathbb{Z}_{12}/\langle 4\rangle)$ and $\Phi(\mathbb{Z}_{12}/\langle 6\rangle)$ are presented in Fig.2, where the negative edges are colored blue and the positive edges are colored black. From these graphs, we see $\langle 4\rangle$ is primary since $\Phi(\mathbb{Z}_{12}/\langle 4\rangle)$ is CN and $\langle 6\rangle$ is semiprime since $\Phi(\mathbb{Z}_{12}/\langle 6\rangle)$ is CP.

Although, from Corollary 3.2, one could determine whether the ideal I is prime, semiprime or primary by investigating the signed zero-divisor graph $\Phi(R/I)$, it is more interesting to find an approach to investigate I just by studying $\Phi(R)$ instead of $\Phi(R/I)$.

Lemma 3.4 *If R is a finite commutative ring and $r \in R \setminus Z(R)$, then there exists a positive integer k such that $r^k = e$.*

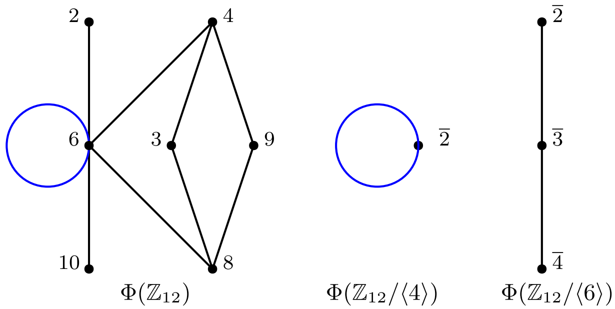

Fig. 2 The signed zero-divisor graphs

Proof For each positive integer k, we have $r^k \in R$. Since $r \in R \setminus Z(R)$, we have $r^k \neq 0$ for any k. Moreover, since R is finite, there exist k_1 and k_2 satisfying $k_1 < k_2$ such that $r^{k_1} = r^{k_2}$. It yields that $r^{k_2-k_1} = e$ because r is not a zero-divisor. □

Lemma 3.5 *Let R be a finite commutative ring and I a non-trivial ideal of R. If $\overline{r} \in Z(R/I)^*$ then $\overline{r} \subseteq Z(R)^*$.*

Proof We only needs to show that $r \in Z(R)^*$ because each element in $\overline{r}$ could be the representative element. Since $\overline{r} \in Z(R/I)^*$, there exists $\overline{r'} \neq \overline{0}$ such that $\overline{rr'} = \overline{0}$. Suppose to the contrary that $r \notin Z(R)$. Lemma 3.4 indicates that there exists a positive integer k such that $r^k = e$. Therefore, $r' = r^k r' = r^{k-1}(rr') \in I$ and thus $\overline{r'} = \overline{0}$, a contradiction. □

From Lemma 3.5, we get the following result.

Lemma 3.6 *Let R be a finite commutative ring and I an ideal of R. If $\overline{r} \in V(\Phi(R/I))$, then $\overline{r} \subseteq V(\Phi(R))$. Furthermore, for two elements $\overline{r}, \overline{r'} \in Z(R/I)^*$, if there is a negative edge between the two vertex sets $\overline{r}$ and $\overline{r'}$ in $\Phi(R)$, then there is a negative edge between the two vertices $\overline{r}$ and $\overline{r'}$ in $\Phi(R/I)$.*

Proof The first part is immediate from Lemma 3.5, and we only prove the second part. If there is a negative edge between the two subsets $\overline{r}$ and $\overline{r'}$ in $\Phi(R)$, then there exist $r_1 \in \overline{r}$ and $r_2 \in \overline{r'}$ satisfying $r_1^{k_1} = r_2^{k_2} = r_1 r_2 = 0$ for some integers k_1 and k_2. Therefore, in R/I, we have $\overline{r}^{k_1} = \overline{r_1}^{k_1} = \overline{0}$, $\overline{r'}^{k_2} = \overline{r_2}^{k_2} = \overline{r_2^{k_2}} = \overline{0}$ and $\overline{r} \cdot \overline{r'} = \overline{r_1} \cdot \overline{r_2} = \overline{r_1 r_2} = \overline{0}$. It indicates that there is a negative edge between $\overline{r}$ and $\overline{r'}$ in $\Phi(R/I)$. □

Lemma 3.6 is useful in some case although it is very simple.

Corollary 3.7 *Let R be a finite commutative ring and I an ideal of R. If there is a negative edge between $r + I$ and $r' + I$ in $\Phi(R)$ for some $r + I, r' + I \in Z(R/I)^*$, then I is not semiprime.*

Motivated by Lemma 3.6, we would like to construct a novel graph from $\Phi(R)$ and I.

Definition 3.8 Let R be a finite commutative ring and I an ideal of R. We construct the new signed graph $\Phi_I(R)$ with vertex set

$$V(\Phi_I(R)) = \{r + I \mid r \in Z(R)^*, r + I \subseteq Z(R)^*\},$$

and the adjacency relations of the vertices are as follows:

- if there is no edge between two distinct vertex sets $r + I$ and $r' + I$ in $\Phi(R)$, then $r + I$ and $r' + I$ are not adjacent in $\Phi_I(R)$;
- if all edges between two distinct vertex sets $r + I$ and $r' + I$ are positive in $\Phi(R)$, then there is a positive edge between $r + I$ and $r' + I$ in $\Phi_I(R)$;
- if there is a negative edge between two distinct vertex sets $r + I$ and $r' + I$ in $\Phi(R)$, then there is a negative edge between $r + I$ and $r' + I$ in $\Phi_I(R)$;

- if there is an edge in the vertex set $r + I$ in $\Phi(R)$, then there is a negative loop incident to $r + I$ in $\Phi_I(R)$.

It is happened to find that $\Phi(\mathbb{Z}_{12}/\langle 4\rangle) = \Phi_{\langle 4\rangle}(\mathbb{Z}_{12})$ and $\Phi(\mathbb{Z}_{12}/\langle 6\rangle) = \Phi_{\langle 6\rangle}(\mathbb{Z}_{12})$. We may call an ideal I of R graph-coincident if $\Phi(R/I) = \Phi_I(R)$, and call the ring R is graph-coincident if all its ideals are graph-coincident. Clearly, the trivial ideals $\{0\}$ and R are graph-coincident. With respect to this notion, Corollary 3.2 could be rewritten as follows.

Corollary 3.9 *Let I be a graph-coincident ideal of the commutative ring R. Then I is prime if and only if $\Phi_I(R)$ is a null graph, is primary if and only if $\Phi_I(R)$ is CN, and is semiprime if and only if $\Phi_I(R)$ is CP.*

Note that $\Phi_I(R)$ could be obtained just by investigating $\Phi(R)$ and the cosets of I but not the algebraic operations in R/I. Corollary 3.9 provides a simpler approach to determine whether I is prime, primary or semiprime provided I is graph-coincident.

4 Adjacency Matrices and Spectra

In this part, we study the adjacency matrices and spectra of signed zero-divisor graphs. In what follows, we denote by $\mathbf{0}_{m,n}$, $\mathbf{J}_{m,n}$ and $\mathbf{1}_n$ the all-zero matrix of size $m \times n$, the all-one matrix of size $m \times n$ and the all-one column vector of size n, respectively. If the sizes of such matrices is evident from the context, we may delete the subscripts for convenience. Note that $\Phi(\mathbb{Z}_p)$ is a null graph for a prime p. We first investigate $\Phi(\mathbb{Z}_{p^m})$ for $m \geq 2$.

Lemma 4.1 *For a prime p and a positive integer $m \geq 2$, the adjacency matrix of $\Phi(\mathbb{Z}_{p^m})$ is*

$$A(\Phi(\mathbb{Z}_{p^m})) = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & -\mathbf{J}_{n_1,n_{m-1}} \\ \mathbf{0} & \mathbf{0} & \cdots & -\mathbf{J}_{n_2,n_{m-2}} & -\mathbf{J}_{n_2,n_{m-1}} \\ \vdots & \vdots & & \vdots & \vdots \\ \mathbf{0} & -\mathbf{J}_{n_{m-2},n_2} & \cdots & -\mathbf{J}_{n_{m-2},n_{m-2}} & -\mathbf{J}_{n_{m-2},n_{m-1}} \\ -\mathbf{J}_{n_{m-1},n_1} & -\mathbf{J}_{n_{m-1},n_2} & \cdots & -\mathbf{J}_{n_{m-1},n_{m-2}} & -\mathbf{J}_{n_{m-1},n_{m-1}} \end{pmatrix},$$

where $n_i = p^{m-i} - p^{m-i-1}$ for $1 \leq i \leq m-1$.

Proof Let $V_i = \{1 \leq x \leq p^m \mid gcd(x, p^m) = p^i\}$ for $1 \leq i \leq m-1$. It is clear that $V_i \cap V_j = \emptyset$ for $i \neq j$ and $Z(\mathbb{Z}_{p^m})^* = \mathrm{Nil}(\mathbb{Z}_{p^m})^* = V_1 \cup \cdots \cup V_{m-1}$. Moreover, $|V_i| = \varphi(p^{m-i})$, where $\varphi(\cdot)$ is the Euler's totient function. For convenience, denote by $n_i = \varphi(p^{m-i})$ for $1 \leq i \leq m-1$.

Now, we may check that, for two vertex sets V_i and V_j (which may be equal), if $i + j \geq m$, then every vertex in V_i is adjacent to all vertices of V_j by negative edges; if $i + j \leq m-1$, then there are no edges between V_i and V_j. Therefore, the adjacency matrix of $\Phi(\mathbb{Z}_{p^m})$ is

$$A(\Phi(\mathbb{Z}_{p^m})) = -\begin{pmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{J}_{n_1,n_{m-1}} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{J}_{n_2,n_{m-2}} & \mathbf{J}_{n_2,n_{m-1}} \\ \vdots & \vdots & & \vdots & \vdots \\ \mathbf{0} & \mathbf{J}_{n_{m-2},n_2} & \cdots & \mathbf{J}_{n_{m-2},n_{m-2}} & \mathbf{J}_{n_{m-2},n_{m-1}} \\ \mathbf{J}_{n_{m-1},n_1} & \mathbf{J}_{n_{m-1},n_2} & \cdots & \mathbf{J}_{n_{m-1},n_{m-2}} & \mathbf{J}_{n_{m-1},n_{m-1}} \end{pmatrix}.$$

□

By using the knowledge of equitable partition [9, Pages 197–198], we get the following result.

Corollary 4.2 *For a prime p and a positive integer $m \geq 2$, the nullity of $\Phi(\mathbb{Z}_{p^m})$ is $p^{m-1} - m$, and the characteristic polynomial of $\Phi(\mathbb{Z}_{p^m})$ is*

$$\phi_{\Phi(\mathbb{Z}_{p^m})}(\lambda) = \lambda^{p^{m-1}-m}\phi_T(\lambda),$$

where $\phi_T(\lambda)$ is the characteristic polynomial of the matrix

$$T = -\begin{pmatrix} 0 & 0 & \cdots & 0 & p-1 \\ 0 & 0 & \cdots & p^2-p & p-1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & p^{m-2}-p^{m-3} & \cdots & p^2-p & p-1 \\ p^{m-1}-p^{m-2} & p^{m-2}-p^{m-1} & \cdots & p^2-p & p-1 \end{pmatrix}.$$

Example 4.3 For a prime p, according to Lemma 4.1 and Corollary 4.2, the adjacency matrix and characteristic polynomial of $\Phi(\mathbb{Z}_{p^2})$ are

$$A(\Phi(\mathbb{Z}_{p^2})) = -\mathbf{J}_{p-1,p-1} \text{ and } \phi_{\Phi(\mathbb{Z}_{p^2})}(\lambda) = \lambda^{p-2}(\lambda + p - 1).$$

Therefore, the spectrum of $\Phi(\mathbb{Z}_{p^2})$ is

$$\mathrm{Sp}(\Phi(\mathbb{Z}_{p^2})) = \{[0]^{p-2}, -(p-1)\}.$$

The adjacency matrix and characteristic polynomial of $\Phi(\mathbb{Z}_{p^3})$ are

$$A(\Phi(\mathbb{Z}_{p^3})) = \begin{pmatrix} \mathbf{0} & -\mathbf{J}_{p^2-p,p-1} \\ -\mathbf{J}_{p-1,p^2-1} & -\mathbf{J}_{p-1,p-1} \end{pmatrix} \text{ and}$$
$$\phi_{\Phi(\mathbb{Z}_{p^3})}(\lambda) = \lambda^{p^2-3}(\lambda^2 + (p-1)\lambda - p(p-1)^2).$$

Therefore, the spectrum of $\Phi(\mathbb{Z}_{p^3})$ is

$$\mathrm{Sp}(\Phi(\mathbb{Z}_{p^3})) = \left\{[0]^{p^2-3}, \pm\frac{(p-1)(\sqrt{4p+1}-1)}{2}\right\}.$$

For $m \geq 4$, we could not find a acceptable formula to express $\phi_T(\lambda)$ even though the matrix T is very clear and neat.

For two signed graph Φ_1 and Φ_2, their product $\Phi_1 \times \Phi_2$ is the signed graph with vertex set $V = V(\Phi_1) \times V(\Phi_2)$, and $(u, v) \sim (u', v')$ if $u \sim u'$ in Φ_1 and $v \sim v'$ in Φ_2. Furthermore, the edge $\{(u, v), (u', v')\}$ is negative if and only if both the edges $\{u, u'\}$ and $\{v, v'\}$ are negative.

Theorem 4.4 *Let R_1 and R_2 be two finite commutative local rings. If A_1 and A_2 are, respectively, the adjacency matrix of $\Phi(R_1)$ and $\Phi(R_2)$, then the adjacency matrix of $\Phi(R_1 \oplus R_2)$ is*

$$A = \begin{pmatrix} A_2 & \mathbf{0} & -\mathbf{J} & \mathbf{1}^T \otimes A_2 & \mathbf{0} & \mathbf{J} & -\mathbf{1}^T \otimes A_2 \\ \mathbf{0} & \mathbf{0} & \mathbf{J} & \mathbf{0} & \mathbf{0} & \mathbf{J} & \mathbf{0} \\ -\mathbf{J} & \mathbf{J} & A_1 & A_1 \otimes \mathbf{1}^T & -\mathbf{1}^T \otimes A_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{1} \otimes A_2 & \mathbf{0} & A_1 \otimes \mathbf{1} & A_1 \otimes A_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{1} \otimes A_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{J} & \mathbf{J} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{1} \otimes A_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix},$$

where $X \otimes Y$ represents the Kronecker product of the matrices X and Y.

Proof For convenience, we denote by

$$\begin{cases} V_1 = \{0\} \oplus Z(R_2)^*, \ V_2 = \{0\} \oplus \overline{Z(R_2)}, \\ V_3 = Z(R_1)^* \oplus \{0\}, \ V_4 = Z(R_1)^* \oplus Z(R_2)^*, \ V_5 = Z(R_1)^* \oplus \overline{Z(R_2)}, \\ V_6 = \overline{Z(R_1)} \oplus \{0\}, \quad V_7 = \overline{Z(R_1)} \oplus Z(R_2)^*. \end{cases}$$

It is easy to verify that $V(\Phi(R_1 \oplus R_2)) = V_1 \cup V_2 \cup \cdots \cup V_7$ is a partition of the vertex set.

Firstly, we discuss the edges between these subsets.

- There are no edges between V_1 and $V_2 \cup V_5$.
- There is a negative edge between any vertex in V_1 and any vertex in V_3.
- For any vertex $(0, r_2)$ in V_1 and any vertex (r_1, r_2') in V_4, if $r_2 \sim r_2'$ in $\Phi(R_2)$, then there is a negative edge between $(0, r_2)$ and (r_1, r_2'); otherwise, they are not adjacent.
- There is a positive edge between any vertex in V_1 and any vertex in V_6.
- For any vertex $(0, r_2)$ in V_1 and any vertex (r_1, r_2') in V_7, if $r_2 \sim r_2'$ in $\Phi(R_2)$, then there is a positive edge between $(0, r_2)$ and (r_1, r_2'); otherwise, they are not adjacent.
- There are no edges between V_2 and $V_4 \cup V_5 \cup V_7$.
- There is a positive edge between any vertex in V_2 and any vertex in $V_3 \cup V_6$.
- There are no edges between V_3 and $V_6 \cup V_7$.
- For any vertex $(r_1, 0)$ in V_3 and any vertex (r_1', r_2) in V_4, if $r_1 \sim r_1'$ in $\Phi(R_1)$, then there is a negative edge between $(r_1, 0)$ and (r_1', r_2); otherwise, they are not adjacent.

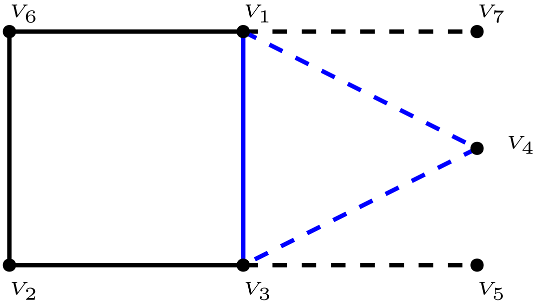

Fig. 3 The diagram of $\Phi(R_1 \oplus R_2)$

- For any vertex $(r_1, 0)$ in V_3 and any vertex (r_1', r_2) in V_5, if $r_1 \sim r_1'$ in $\Phi(R_1)$, then there is a positive edge between $(r_1, 0)$ and (r_1', r_2); otherwise, they are not adjacent.
- There are no edges between V_i and V_j where $\{i, j\} \subset \{4, 5, 6, 7\}$.

To make the structure more clear, we present it in Fig.3. In this figure, each vertex represents a subset, the black (blue) fat line between two parts means that every vertex in one part is adjacent to every vertex in the other part by a positive (resp. negative) edge, and the black (blue) dashed line between two parts means that there is positive (resp. negative) edge between one vertex (r_1, r_2) in one part and another vertex (r_1', r_2') in the other part if and only if either $r_1 \sim r_1'$ in $\Phi(R_1)$ or $r_2 \sim r_2'$ in $\Phi(R_2)$.

Next, we discuss the edges in every subsets.

- The subset V_1 induces a copy of $\Phi(R_2)$, V_3 induces a copy of $\Phi(R_1)$, and V_4 induces a copy of $\Phi(R_1) \times \Phi(R_2)$.
- The subsets V_2, V_5, V_6, V_7 are all independent sets.

At last, the adjacency matrix of $\Phi(R_1 \oplus R_2)$ is obtained, which is

$$A = \begin{pmatrix} A_2 & \mathbf{0} & -\mathbf{J} & \mathbf{1}^T \otimes A_2 & \mathbf{0} & \mathbf{J} & -\mathbf{1}^T \otimes A_2 \\ \mathbf{0} & \mathbf{0} & \mathbf{J} & \mathbf{0} & \mathbf{0} & \mathbf{J} & \mathbf{0} \\ -\mathbf{J} & \mathbf{J} & A_1 & A_1 \otimes \mathbf{1}^T & -\mathbf{1}^T \otimes A_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{1} \otimes A_2 & \mathbf{0} & A_1 \otimes \mathbf{1} & A_1 \otimes A_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{1} \otimes A_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{J} & \mathbf{J} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{1} \otimes A_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix},$$

here we omit all the subscripts of the blocks in A as their sizes are clear. □

Remark 4.5 Theorem 4.4 gives the specific structure of $\Phi(R_1 \oplus R_2)$ when both R_1 and R_2 are local. However, when one of them is not local, the case will be very fussy.

Example 4.6 Let p and q be two primes which may be equal. Note that $\Phi(\mathbb{Z}_p)$ is a null graph. According to Example 4.3 and Theorem 4.4, the adjacency matrix and characteristic polynomial of $\Phi(\mathbb{Z}_p \oplus \mathbb{Z}_{q^2})$ are

$$A=\begin{pmatrix} -\mathbf{J}_{q-1,q-1} & \mathbf{0}_{q-1,q^2-q} & \mathbf{J}_{q-1,p-1} & \mathbf{1}_{p-1}^T\otimes\mathbf{J}_{q-1,q-1} \\ \mathbf{0}_{q^2-q,q-1} & \mathbf{0}_{q^2-q,q^2-q} & \mathbf{J}_{q^2-q,p-1} & \mathbf{0}_{q^2-q,(p-1)(q-1)} \\ \mathbf{J}_{p-1,q-1} & \mathbf{J}_{p-1,q^2-q} & \mathbf{0}_{p-1,p-1} & \mathbf{0}_{p-1,(p-1)(q-1)} \\ \mathbf{1}_{p-1}\otimes\mathbf{J}_{q-1,q-1} & \mathbf{0}_{(p-1)(q-1),q^2-q} & \mathbf{0}_{(p-1)(q-1),p-1} & \mathbf{0}_{(p-1)(q-1),(p-1)(q-1)} \end{pmatrix}$$

and $\phi_{\Phi(\mathbb{Z}_p\oplus\mathbb{Z}_{q^2})}(\lambda)=\lambda^{q^2+(p-1)q-5}f(\lambda)$, where

$$f(\lambda)=\lambda^4+(q-1)\lambda^3-2q(p-1)(q-1)\lambda^2-q(p-1)(q-1)^2\lambda+q(p-1)^2(q-1)^3.$$

Therefore, the spectrum of $\Phi(\mathbb{Z}_p\oplus\mathbb{Z}_{q^2})$ is

$$\mathrm{Sp}(\Phi(\mathbb{Z}_p\oplus\mathbb{Z}_{q^2}))=\left\{[0]^{q^2+(p-1)q-5},\lambda_1,\lambda_2,\lambda_3,\lambda_4\right\},$$

where $\lambda_1,\lambda_2,\lambda_3,\lambda_4$ are roots of $f(\lambda)$.

Similarly, one could obtain the adjacency matrices and characteristic polynomials of $\Phi(\mathbb{Z}_{p^i}\oplus\mathbb{Z}_{q^j})$ for $1\le i,j\le 3$.

5 Conclusion

In this paper, we associate a signed graph with a commutative ring, that is, the signed zero-divisor graph. By investigating this signed graph, we try to explore the structure of the ring from different points of view. Since a signed zero-divisor graph contains both positive edges and negative edges, we define the positive subgraph and negative subgraph. We first investigate the distances in the positive and negative subgraphs, from which we get some results on the corresponding signed zero-divisor graphs and the classical zero-divisor graphs (which is called reduced zero-divisor graphs in this paper). We also consider whether a reduced signed zero-divisor graph is balanced or not. In fact, we completely solve this problem on the ring $\mathbb{Z}_n$ for any n. However, for general rings, this problem is far from being solved. So we propose

Problem 5.1 *For a commutative ring R, give a sufficient and necessary condition, in terms of the property of R, for $\tilde{\Phi}(R)$ to be balanced.*

Next, we try to study the ideals of a ring R by investigating the signed zero-divisor graph $\Phi(R)$. There are relations between the properties of an ideal I of the ring R and the signed zero-divisor graph $\Phi(R/I)$. However, we would like to investigate I but avoid studying the operations in R/I. Therefore, we construct a new signed graph $\Phi_I(R)$, and find some connections between $\Phi(R/I)$ and $\Phi_I(R)$. Interestingly, for some ideal I, these two signed graphs could be the same. Hence, we define the graph-coincident ideal to be the ideals satisfying $\Phi_I(R)=\Phi(R/I)$. We know little about such ideals and leave the following problems for further research.

Problem 5.2 *Let R be a finite commutative ring and I an ideal of R. Give a sufficient and necessary condition for I to be graph-coincident.*

Problem 5.3 *Let R be a finite commutative ring. Give a sufficient and necessary condition for R to be graph-coincident.*

At last, we study the adjacency matrices and spectra of signed zero-divisor graphs. In fact, we completely determine the adjacency matrix of $\Phi(R_1 \oplus R_2)$, in terms of the adjacency matrices of $\Phi(R_1)$ and $\Phi(R_2)$, when R_1 and R_2 are local. This means that the structure of $\Phi(R_1 \oplus R_2)$ is clear whenever the graphs $\Phi(R_1)$ and $\Phi(R_2)$ are given. Note that any finite commutative ring could be expressed as the direct sum of local rings, that is, $R = R_1 \oplus R_2 \oplus \cdots \oplus R_n$. Therefore, we could determine the adjacency matrix of $\Phi(R)$ by induction on n if we could determine the adjacency matrix of $\Phi(R_1 \oplus R_2)$ whenever R_1 is local but R_2 is not. However, when R_2 is not local, we could not partition $V(\Phi(R_1 \oplus R_2))$ into seven subsets as we did in the proof of Theorem 4.4, because we should distinguish the nilpotent elements and other zero-divisors of R_2. It will be much more complicated. A concise method to determine the adjacency matrix of $\Phi(R)$ is needed. It would be a good start to determine the adjacency matrix and the spectrum of $\Phi(\mathbb{Z}_n)$.

Problem 5.4 *Determine the adjacency matrix and the spectrum of $\Phi(\mathbb{Z}_n)$.*

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