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Mixed graphs with smallest eigenvalue greater than $-\sqrt{3}$

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In this paper, we determine the underlying graphs of the mixed graphs with smallest Hermitian eigenvalue greater than $-\sqrt{3}$. Furthermore, we characterize all mixed graphs on $n \geq 11$ vertices with smallest Hermitian eigenvalue greater than $-\sqrt{3}$. The mixed graphs on $n \leq 10$ vertices with smallest Hermitian eigenvalue greater than $-\sqrt{3}$ could be also obtained easily with the help of computer because their underlying graphs are determined. Roughly speaking, we completely determine the mixed graphs with smallest Hermitian eigenvalue greater than $-\sqrt{3}$.

1. Introduction

It is a classical problem in Spectral Graph Theory to characterize the graphs whose eigenvalues are bounded. The research of such problems may date back to the work of Smith in 1970 [14]. This work stimulated the interest of researchers. There are a lot of results in the literature concerning the topic. In 1972, Hoffman [8] obtained all limit points of the spectral radii of non-negative symmetric matrices smaller than $\frac{\sqrt{5}+1}{2}$.

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In 1982, Cvetković et al. [2] characterized the graphs whose spectral radius does not exceed $\sqrt{2+\sqrt{5}}$ and in 1989, Brouwer and Neumaier [1] determined the graphs with spectral radius between 2 and $\sqrt{2+\sqrt{5}}$ and later, Woo and Neumaier [18] described the structure of graphs whose spectral radius are bounded above by $3\sqrt{2}/2$. With respect to the smallest eigenvalues, Hoffman [9] investigated the graphs whose smallest eigenvalue exceeds $-1-\sqrt{2}$, and this work was continued by Taniguchi et al. [15,16,10]. Especially, the graphs with smallest eigenvalue -2 attracted a lot of attention, and we refer the reader to the survey [3] and the book [4]. In this paper we consider the smallest Hermitian eigenvalues of mixed graphs.

A mixed graph is defined to be an ordered triple (V, E, A), where V is the vertex set, E is the undirected edge set and A is the directed edge set. Note that, if both uv and vu are directed edges, then we regard $\{u, v\} \in E$ as an undirected edge. Thus, if $(u, v) \in A$ then $(v, u) \notin A$. Clearly, if $A = \emptyset$ then the mixed graph turns to be a graph and if $E = \emptyset$ then the mixed graph turns to be an oriented graph. For convenience, we write $u \leftrightarrow v$ if $\{u, v\} \in E$ and $u \to v$ if $(u, v) \in A$. If $v_1 \leftrightarrow v_2$ for any $v_1 \in V_1$, then we write $V_1 \leftrightarrow v_2$. We could define the notations $v_2 \to V_1$ and $V_1 \to v_2$ similarly. If $v_1 \leftrightarrow v_2$ for any $v_1 \in V_1$ and $v_2 \in V_2$, then we write $V_1 \leftrightarrow V_2$. We could also define the notation $V_1 \to V_2$ similarly. Moreover, we say v and v' are *twins with respect to the vertex* u, denoted by $v \sim_u v'$, if, $v \leftrightarrow u$ if and only if $v' \leftrightarrow u$, $v \to u$ if and only if $v' \to u$, and $u \to v$ if and only if $u \to v'$. If the vertex set $W \subseteq V$ satisfies $w \sim_u w'$ for any two vertices $w, w' \in W$, then W is a *twin set with respect to* u, denoted by $W_{\sim_u}$. If $W_{\sim_u}$ for any $u \in U$ then we write $W_{\sim_U}$. If $W_{\sim_U}$ and $U_{\sim_W}$ then we write $W \asymp U$.

Let $M = (V, E, A)$ be a mixed graph with $V = \{v_1, v_2, \ldots, v_n\}$. The *underlying graph* $\Gamma(M)$ is a graph with vertex set V and two vertices $u \sim v$ if either $u \leftrightarrow v$, $u \to v$ or $v \to u$. For $U \subseteq V$ and $W \subseteq V \setminus U$, denote by $N_W(U) = \{w \mid w \in W, u \sim w \text{ in } \Gamma(M) \text{ for some } u \in U\}$. Especially, if $U = \{u\}$ then $N_W(u)$ is the set of neighbors of u in W. Moreover, denote by $N^+_W(u) = \{w \mid u \to w\}$, $N^-_W(u) = \{w \mid u \leftarrow w\}$ and $N^o_W(u) = \{w \mid u \leftrightarrow w\}$. It is clear that $N_W(u) = N^+_W(u) \cup N^-_W(u) \cup N^o_W(u)$. As usual, we always write P_n, C_n, $K_{n_1,n_2,\ldots,n_k}$ the path, the cycle and the complete multipartite graph of the corresponding order. For two graphs G and H, the *union* $G \cup H$ is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. The *join* $G \nabla H$ is the graph obtained from $G \cup H$ by adding all edges between G and H. The *distance* of two vertices $u, v \in V(G)$ in G is the length of a shortest path from u to v in G, denoted by $d_G(u, v)$. The *diameter* of G is the largest distance in G, denoted by $d(G)$. All other notations not mentioned here are standard in [5].

We always write M_G for M when the underlying graph $\Gamma(M) = G$. Moreover, for a graph G, denote by $\mathcal{M}_G$ the set of mixed graphs with underlying graph G. Especially, if $M_G = G$ then we write G for M_G. The mixed graph M_G is *connected* if G is connected and we always consider the connected mixed graphs in this paper. The *diameter* of M_G is defined to be the diameter of G, denoted by $d(M_G)$. For a subset $U \subseteq V$, the mixed subgraph induced by U is the mixed graph $M_G[U] = (U, E', A')$ with $E' = \{\{u, v\} \mid$

$u, v \in U, \{u, v\} \in E\}$ and $A' = \{(u, v) \mid u, v \in U, (u, v) \in A\}$. The *Hermitian matrix* of M_G is defined to be a square matrix $H(M_G) = [h_{st}]_{n\times n}$ with

$$h_{st} = \begin{cases} 1, & v_s \leftrightarrow v_t, \\ i, & v_s \to v_t, \\ -i, & v_t \to v_s, \\ 0, & \text{otherwise}, \end{cases}$$

which was proposed by Liu and Li [11] and Guo and Mohar [6] independently. Since $H(M_G)$ is a Hermitian matrix, all eigenvalues of $H(M_G)$ are real and listed as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The collection of such eigenvalues is the spectrum of $H(M_G)$. The *Hermitian spectrum* of the mixed graph M_G is just the spectrum of $H(M_G)$, denoted by $\mathrm{Sp}(M_G)$. For convenience, we always write $\lambda(M_G)$ for the smallest eigenvalue of M_G. Two mixed graphs $M_G, M'_G \in \mathcal{M}_G$ are *switching equivalent* if there exists a diagonal matrix D whose entries belong to $\{\pm 1, \pm i\}$ such that $H(M'_G) = DH(M_G)D^*$. It is clear that the relation switching equivalence is an equivalence relation. Thus, denote by $[M_G]$ the equivalence class containing M_G with respect to switching equivalence. Obviously, all graphs in $[M_G]$ share the same spectrum. In 2017, Guo and Mohar [7] determined all mixed graphs with Hermitian spectral radius below 2. Recently, Yuan et al. [19] characterized all mixed graphs with Hermitian spectral radius at most 2 when G contains no cycles of length 4 and Lu [12] determined the connected mixed graphs with smallest Hermitian eigenvalue greater than $-\frac{\sqrt{5}+1}{2}$.

In this paper, we try to characterize the connected mixed graphs with smallest Hermitian eigenvalue greater than $-\sqrt{3}$. Denote by $\mathcal{M}_n(-\sqrt{3})$ the set of connected mixed graphs on n vertices with $\lambda > -\sqrt{3}$. We first get the underlying graphs of the graphs in $\mathcal{M}_n(-\sqrt{3})$. Next, we determine $\mathcal{M}_n(-\sqrt{3})$ for $n \geq 11$. For $n \leq 10$, since the underlying graphs of the mixed graphs $\mathcal{M}_n(-\sqrt{3})$ are obtained, the mixed graphs $\mathcal{M}_n(-\sqrt{3})$ could be obtained with the help of computer immediately.

2. Preliminaries

In this part, we will introduce some results which will be used latter. We first present the famous interlacing theorem with respect to Hermitian matrix.

Lemma 1 *([11]). Let M_G be a mixed graph with underlying graph G. If M_H is a mixed induced subgraph of M_G, then the eigenvalues of M_H interlace those of M_G.*

Next we introduce another powerful tool in spectral graph theorem, that is the equitable partition. Let M_G be a mixed graph on n vertices with underlying graph G. Let π: $V(G) = V_1 \cup V_2 \cup \cdots \cup V_s$ be a partition of $V(G)$ with $|V_k| = n_k$ and $n = n_1 + n_2 + \cdots + n_s$. For $1 \leq k, l \leq s$, denote by H_{kl} the submatrix of $H(M_G)$ whose rows corresponding to V_k and columns corresponding to V_l. Therefore, the Hermitian matrix $H(M_G)$ can be written as $H(M_G) = [H_{kl}]$. Denote by $b_{kl} = \mathbf{1}^T H_{kl}\mathbf{1}/n_k$ the average row-sums of H_{kl}, where

1 denotes the all-one vector. The matrix $H_\pi = [b_{kl}]_{s\times s}$ is called the *quotient matrix* of $H(M_G)$. If, for any k, l, the row-sum of H_{kl} corresponding to any vertex $v \in V_k$ equals to b_{kl}, then π is called an *equitable partition* of M_G. Let δ_{V_k} be a vector indexed by $V(G)$ such that $\delta_{V_k}(v) = 1$ if $v \in V_k$ and 0 otherwise. The matrix $P = [\delta_{V_1}\delta_{V_2}\cdots\delta_{V_s}]$ is called the *characteristic matrix* of π. If π is an equitable partition, then $H(M_G)P = PH_\pi$. It leads to the following famous result.

Lemma 2 *([5, Theorem 9.3.3, page 197]). Let M_G be a mixed graph and π an equitable partition of M_G with quotient matrix H_π and characteristic matrix P. Then the eigenvalues of H_π are also eigenvalues of $H(M_G)$. Furthermore, $H(M_G)$ has the following two kinds of eigenvectors:*

(i) *the eigenvectors in the column space of P, and the corresponding eigenvalues coincide with the eigenvalues of H_π;*
(ii) *the eigenvectors orthogonal to the columns of P, i.e., those eigenvectors sum to zero on each cell of π.*

Let $\mathcal{H}$ be a set of graphs. A graph G is called $\mathcal{H}$-free if none of graphs in $\mathcal{H}$ can be an induced subgraph of G. Especially, if $\mathcal{H} = \{H\}$ then the $\mathcal{H}$-free graph G is also called an H-free graph. Recall that a P_4-free graph is called a cograph. The following result reveals the structure of cographs.

Lemma 3 *([13]). If G is a connected P_4-free graph, then G is the join of two graphs, that is, $G = G_1\nabla G_2$ for some graphs G_1 and G_2 with $|V(G_1)|, |V(G_2)| \geq 1$.*

We determine some types of $\mathcal{H}$-free graphs when $\mathcal{H}$ contains some simple graphs.

Lemma 4. *If G is a $\{2K_1, K_3\}$-free graph then $G \in \{K_1, K_2\}$. If G is $\{2K_1, K_4\}$-free graph then $G \in \{K_1, K_2, K_3\}$.*

Proof. It is clear that, if a graph G is $2K_1$-free, then it is a complete graph. Thus, we have $G \in \{K_1, K_2\}$ if G is additional K_3-free and $G \in \{K_1, K_2, K_3\}$ if G is additional K_4-free. □

Lemma 5. *If G is a $\{2K_2, K_3, 3K_1, K_{1,2}\}$-free graph then $G \in \{K_1, K_2, 2K_1, K_1 \cup K_2\}$.*

Proof. It is clear that, if a graph G is $\{3K_1, K_{1,2}\}$-free, then it is a union of at most two complete graphs. Thus, we have $G \in \{K_1, K_2, 2K_1, K_1 \cup K_2\}$ if G is additional $\{2K_2, K_3\}$-free. □

Lemma 6. *If G is a $\{3K_1, K_1 \cup K_{1,2}, 2K_2, K_1 \cup K_3, P_4, K_1\nabla(K_1 \cup K_2), K_{2,2}, K_2\nabla K_{1,2}\}$-free graph then $G \in \{2K_1, K_1 \cup K_2, K_{1,2}, K_1\nabla K_{1,2}, K_n\}$.*

Proof. It is clear that, if a graph G is $3K_1$-free, then it has at most two connected components. Suppose that G has two connected components X and Y. Since G is $K_1 \cup K_{1,2}$-free, one of X and Y is a complete graph. Combining with that G is $\{2K_2, K_1 \cup K_3\}$-free, we have $G \in \{2K_1, K_1 \cup K_2\}$. Now we suppose G is connected. If $|V(G)| \leq 3$, then $G \in \{K_1, K_2, K_3, K_{1,2}\}$. Note that all connected graphs on 4 vertices are $P_4, K_{1,3}, K_{2,2}, K_1\nabla(K_1 \cup K_2), K_1\nabla K_{1,2}$ and K_4. If $|V(G)| = 4$, then $G = K_1\nabla K_{1,2}$ or K_4 since G is $\{3K_1, P_4, K_{2,2}, K_1\nabla(K_1 \cup K_2)\}$-free.

Next, we show that $|V(G)| \leq 4$ when $G \neq K_n$. Suppose to the contrary that $G \neq K_n$ and $|V(G)| \geq 5$. Therefore, G has a connected induced subgraph H on 4 vertices satisfying $H \neq K_4$. Since H is also $\{3K_1, P_4, K_{2,2}, K_1\nabla(K_1 \cup K_2)\}$-free, we have $H = K_1\nabla K_{1,2} = K_2\nabla 2K_1$. Assume that $V(H) = \{u_1, v_1, u_2, v_2\}$ such that $H[v_1, v_2] = 2K_1$ and $H[u_1, u_2] = K_2$. Let $x \in V(G) \setminus V(H)$ be a vertex adjacent to some vertices of H. If $|N_H(x)| = 4$ then $G[V(H) \cup \{x\}] = K_2\nabla K_{1,2}$, a contradiction. If $|N_H(x)| = 3$ then $G[x, v_1, v_2, u_1] = K_{2,2}$ when $N_H(x) = \{v_1, v_2, u_2\}$, and $G[x, v_1, v_2, u_2] = K_1\nabla(K_1 \cup K_2)$ when $N_H(x) = \{u_1, u_2, v_2\}$, which are both impossible. If $|N_H(x)| = 2$ then $G[x, v_1, v_2] = 3K_1$ when $N_H(x) = \{u_1, u_2\}$, $G[x, v_1, v_2, u_1] = K_{2,2}$ when $N_H(x) = \{v_1, v_2\}$, and $G[x, u_1, v_1, v_2] = K_1\nabla(K_1 \cup K_2)$ when $N_H(x) = \{u_1, v_1\}$, which are all impossible. If $|N_H(x)| = 1$ then $G[x, v_1, v_2] = 3K_1$ when $N_H(x) = \{u_1\}$, and $G[x, v_1, u_2, v_2] = P_4$ when $N_H(x) = \{v_1\}$, which are both impossible.

The proof is completed. □

Lemma 7. *If G is a $\{3K_1, K_{1,2}, K_5\}$-free graph then $G \in \{K_1, K_2, K_3, K_4, 2K_1, K_1 \cup K_2, K_1 \cup K_3, K_1 \cup K_4, 2K_2, K_2 \cup K_3, K_2 \cup K_4, 2K_3, K_3 \cup K_4, 2K_4\}$.*

Proof. It is clear that, if a graph G is $\{K_{1,2}, 3K_1\}$-free, then it is the union of at most two complete graphs. Thus, we have $G \in \{K_1, K_2, K_3, K_4, 2K_1, K_1 \cup K_2, K_1 \cup K_3, K_1 \cup K_4, 2K_2, K_2 \cup K_3, K_2 \cup K_4, 2K_3, K_3 \cup K_4, 2K_4\}$ if G is additional K_5-free. □

Guo and Mohar introduced the so called four-way switching to generate switching equivalent graphs [6]. A *four-way switching* is the operation of changing a mixed graph M_G into the mixed graph M'_G by choosing an appropriate diagonal matrix S with $S_{jj} \in \{\pm 1, \pm i\}$ and setting $H(M'_G) = S^{-1}H(M_G)S$. Let G be a graph and X an edge cut such that $G - X = G_1 \cup G_2$. Define $X^+ = \{(v_1, v_2) \mid \{v_1, v_2\} \in X, v_1 \in V_1, v_2 \in V_2\}$ and $X^- = \{(v_2, v_1) \mid \{v_1, v_2\} \in X, v_1 \in V_1, v_2 \in V_2\}$. The cut X is called a *coincident cut* of the mixed graph $M_G = (V, E, A)$ if either $X^+ \subseteq A$, $X^- \subseteq A$ or $X \subseteq E$. If X is a coincident cut of M_G, then the *X-switching* of M_G is the mixed graph $M_G(X) = (V, E', A')$ with $E' = E \cup X$ and $A' = A \setminus (X^+ \cup X^-)$. Note that $M_G(X) = M_G$ if $X \subseteq E$. From four-way switching, the following results are obtained.

Lemma 8 *([6]). Let M_G be a mixed graph. If X is a coincident cut of M_G, then M_G and $M_G(X)$ are switching equivalent and thus* $\mathrm{Sp}(M_G) = \mathrm{Sp}(M_G(X))$.

If G is a forest, then each edge is a cut. Moreover, each edge is a coincident cut of any mixed graph M_G. Thus, Lemma 8 implies the following result.

Corollary 1 *([6]). If G is a forest, then* $\mathrm{Sp}(M_G) = \mathrm{Sp}(G)$ *for any mixed graph* $M_G \in \mathcal{M}_G$.

Let M be a connected mixed graph and $H(M) = [h_{st}]$ be its Hermitian matrix. The *value* of a mixed walk $W = v_1v_2v_3 \cdots v_l$ is $h(W) = h_{12}h_{23} \cdots h_{(l-1)l}$. A mixed walk is *positive* or *negative* if $h(W) = 1$ or $h(W) = -1$, respectively. Note that for one direction the value of a mixed walk or a mixed cycle is α, then for the reversed direction its value is $\overline{\alpha}$. Thus, if the value of a cycle is 1 (resp. -1) in a direction, then its value is 1 (resp. -1) for the reversed direction. In these situations, we just terms this mixed cycle as a positive (resp. negative) mixed cycle without mentioning any direction. A graph is *positive* (resp. *negative*) if each its mixed cycle is positive (resp. negative).

Lemma 9 *([11]). Let M_G be a mixed graph with the underlying graph G. Then M_G is positive if and only if M_G can be obtained from G by a four-way switching.*

Let G_1 and G_2 be two subgraphs of G. The *symmetric difference* of G_1 and G_2, denoted by $G_1 \oplus G_2$, is defined to be a subgraph of G induced by the symmetric difference of $E(G_1)$ and $E(G_2)$. Let C and C' be two cycles of G. It is easy to see that $C \oplus C'$ is an even graph. If the intersection of C and C' is exactly a non-trivial path $P_k (k \geq 2)$, we say that cycle $C \oplus C'$ is obtained from C and C' by making a *strong symmetric difference*, and write $C \oplus_s C'$ instead of $C \oplus C'$.

Lemma 10 *([17, Lemma 16]). Let C, C^1, C^2 be three cycles of a simple graph G satisfying $C = C^1 \oplus_s C^2$. Under any given orientation of G, if $h(C^1) = h(C^2) = 1$ then $h(C) = 1$.*

3. Mixed graphs with $\boldsymbol{\lambda > -\sqrt{3}}$

Denote by $\mathcal{M}_n(-\sqrt{3})$ the set of connected mixed graphs on n vertices with smallest Hermitian eigenvalue greater than $-\sqrt{3}$. In this part, we first get the underlying graphs of mixed graphs in $\mathcal{M}_n(-\sqrt{3})$ by investigating various mixed induced subgraphs such as mixed triangles, mixed quadrangles and mixed pentagons and so on. Next, we completely determine $\mathcal{M}_n(-\sqrt{3})$ for $n \geq 11$.

Lemma 11 *([12]). Let M_G be a mixed graph with smallest eigenvalue λ. If $\lambda > -\sqrt{3}$, then any mixed triangle in M_G belongs to $\{K_3, K_3^{2,2}, K_3^{2,3}\}$. If $\lambda \geq -1.84$, then any induced mixed quadrangle in M_G belongs to $\{C_4^1, C_4^2, C_4^3\}$ (Fig. 1).*

As similar to Lemma 11, by immediate calculating the Hermitian spectrum of mixed pentagons, one could easily verify that there are 10 types of mixed pentagons satisfied $\lambda(M_{C_5}) > -\sqrt{3}$, and we present them in Fig. 2 together with their smallest eigenvalues.

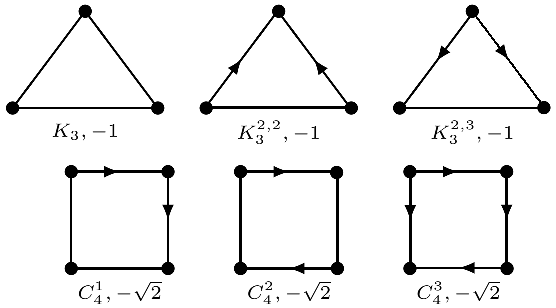

Fig. 1. The mixed triangles and quadrangles and their smallest eigenvalues of Lemma 11.

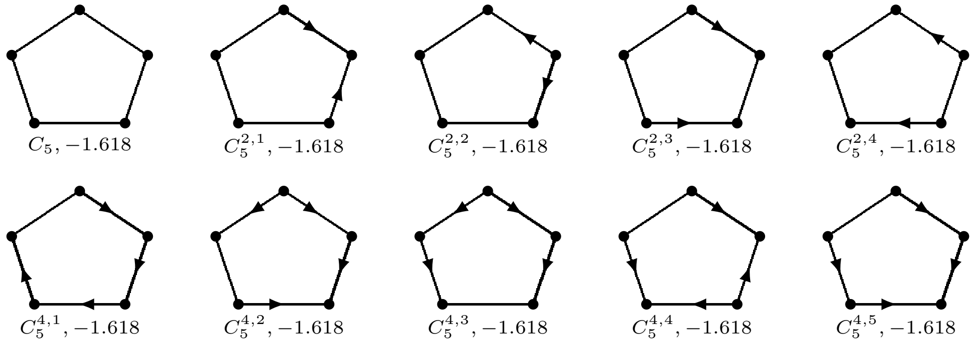

Fig. 2. The mixed pentagons and their smallest Hermitian eigenvalues of Lemma 12.

Lemma 12. *Let M_G be a mixed graph with smallest eigenvalue λ. If $\lambda > -\sqrt{3}$, then any mixed induced pentagon in M_G belongs to $\{C_5, C_5^{2,1}, C_5^{2,2}, C_5^{2,3}, C_5^{2,4}, C_5^{4,1}, C_5^{4,2}, C_5^{4,3}, C_5^{4,4}, C_5^{4,5}\}$.*

In what follows, we always denote $\mathcal{C}_3 = \{K_3, K_3^{-2,2}, K_3^{-2,3}\}$, $\mathcal{C}_4 = \{C_4^1, C_4^2, C_4^3\}$ and $\mathcal{C}_5 = \{C_5, C_5^{2,1}, C_5^{2,2}, C_5^{2,3}, C_4^{2,4}, C_5^{4,1}, C_5^{4,2}, C_5^{4,3}, C_5^{4,4}, C_5^{4,5}\}$. The mixed triangles K_3, $K_3^{2,2}$ and $K_3^{2,3}$ play an important role in determining the orientations of a mixed graph, especially when all induced cycles (if exist) of the underlying graph are triangles.

Remark 1. *It is easy to see that $\mathcal{C}_3 = [C_3]$ and $\mathcal{C}_5 = [C_5]$, which are all positive. Moreover, all mixed graphs in $\mathcal{C}_4$ are negative.*

By Lemma 9, we immediately have the following result.

Lemma 13. *Let M_G be a mixed graph. If G contains no induced cycle with length greater than 3 and each mixed triangle of M_G belongs to $\mathcal{C}_3$ then $M_G \in [G]$.*

For non-negative integers s, t, n with $n = s + t$, denote by $K_n[s, t]$ the mixed graph obtained from $K_s \cup K_t$ by adding all arcs from the vertices of K_s to those of K_t. It is clear that $K_n[s, t]$ is switching equivalent to K_n. Let M_{K_n} be a mixed graph with underlying graph K_n and $n \geq 3$. Lemma 9 indicates that any mixed triangle of M_{K_n} belongs to $\mathcal{C}_3$ if and only if $M_{K_n} \in [K_n]$. Furthermore, according to [6, Proposition 8.6], the following result holds.

Lemma 14 *([6]). Let M_{K_n} be a mixed graph with underlying graph K_n and $n \geq 3$. Then any mixed triangle of M_{K_n} belongs to $\mathcal{C}_3$ if and only if $M_{K_n} \in \{K_n[s,t] \mid s,t \geq 0, s+t = n\}$ if and only if $M_{K_n} \in [K_n]$.*

In what follows, we characterize the mixed graph M_G with $\lambda(M_G) > -\sqrt{3}$. We first determine the underlying graph of M_G.

Lemma 15. *If M_G is a mixed graph with underlying graph $G = K_{m,n}$, then $\lambda(M_G) \leq -\sqrt{3}$ except for $G = K_2, K_{1,2}$ or $K_{2,2}$.*

Proof. If $\lambda(M_G) > -\sqrt{3}$, then G has no $K_{1,3}$ as an induced subgraph since $\lambda_4(K_{1,3}) = -\sqrt{3}$. This leads to $G = K_2, K_{1,2}$ or $K_{2,2}$. $\square$

By applying Lemma 11 and Lemma 13, we get the following result.

Lemma 16. *If $M_G \in \mathcal{M}_n(-\sqrt{3})$ then G is $\mathcal{H}$-free, where*

$$\mathcal{H} = \{\, P_5, K_{1,3}, K_2\nabla 3K_1, K_2\nabla(K_1 \cup K_{1,2}), K_2\nabla 2K_2, K_2\nabla(K_1 \cup K_3),$$
$$K_3\nabla(K_1 \cup K_2), K_4\nabla K_{1,2}, K_1\nabla K_{2,2} = 2K_1\nabla K_{1,2}, F \,\}$$

and F is the graph obtained from $K_2\nabla 2K_1$ by attaching a pendent vertex to a vertex of degree 2 (see Fig. 4).

Proof. We prove this statement by contradiction. Suppose to the contrary that G contains an induced graph H in $\mathcal{H}$. Therefore, M_G contains a mixed induced graph M_H. Lemma 1 indicates that $\lambda(M_H) \geq \lambda(M_G) > -\sqrt{3}$. Note that $\lambda(P_5) = \lambda(K_{1,3}) = -\sqrt{3}$, $\lambda(K_2\nabla 3K_1) = -2$, $\lambda(K_2\nabla(K_1 \cup K_{1,2}) = -2$, $\lambda(K_2\nabla 2K_2) = -1.828$, $\lambda(K_2\nabla(K_1 \cup K_3)) = -1.828$, $\lambda(K_3\nabla(K_1 \cup K_2)) = -1.804$, $\lambda(K_4\nabla K_{1,2}) = -1.742$ and $\lambda(F) = -1.7757$, which are all not greater than $-\sqrt{3}$. It yields that $H = K_1\nabla K_{2,2}$, and we label H as Fig. 3. Since $\lambda(M_H) > -\sqrt{3}$, Lemma 11 implies all mixed triangles of M_H belong to $\mathcal{C}_3$ and all quadrangles of M_H belong to $\mathcal{C}_4$. Let $C^1 = M_H[u_1, u_2, v]$, $C^2 = M_H[u_2, u_3, v]$, $C^3 = M_H[u_3, u_4, v]$ and $C^4 = M_H[u_1, u_4, v]$. Note that the quadrangle with vertices u_1, u_2, u_3, v, denoted by C^5, satisfies $C^5 = C^1 \oplus_s C^2$, and the quadrangle with vertices u_1, u_4, u_3, v, denoted by C^6, satisfies $C^6 = C^3 \oplus_s C^4$ (see Fig. 3). Since C^i is positive for $1 \leq i \leq 4$, Lemma 10 indicates that C^5 and C^6 are also positive. Using Lemma 10 again, $M_H[u_1, u_2, u_3, u_4] = C^5 \oplus_s C^6$ is a positive induced quadrangle, contradicts Lemma 11. $\square$

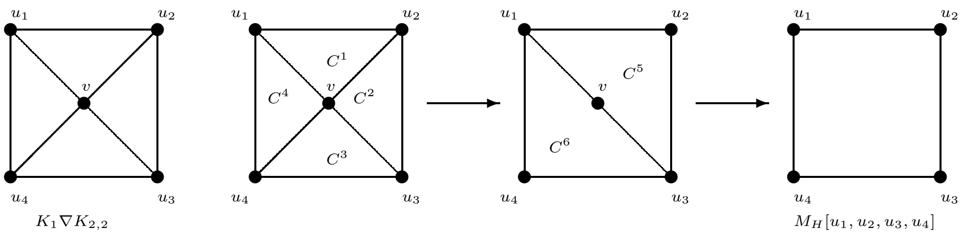

Fig. 3. The graphs used in Lemma 16.

For a mixed graph $M_G \in \mathcal{M}_n(-\sqrt{3})$, we divide into two cases to determine the underlying graph G, that is, G is P_4-free or G is not P_4-free.

Lemma 17. *If $M_G \in \mathcal{M}_n(-\sqrt{3})$ and G is P_4-free then G belongs to*

$$\{K_{2,2}, K_1\nabla K_{1,2}, K_2\nabla K_{1,2}, K_3\nabla K_{1,2}, K_2\nabla(K_1\cup K_2), 2K_2\nabla 2K_1, 2K_3\nabla 2K_1, 2K_4\nabla 2K_1,$$
$$(K_1\cup K_2)\nabla 2K_1, (K_1\cup K_3)\nabla 2K_1, (K_1\cup K_4)\nabla 2K_1, (K_2\cup K_3)\nabla 2K_1, (K_2\cup K_4)\nabla 2K_1,$$
$$(K_3\cup K_4)\nabla 2K_1\}\bigcup\{(K_s\cup K_t)\nabla K_1 \mid s,t\geq 0, s+t=n-1\}$$

Proof. We may assume that $n \geq 2$ since there are nothing to prove when $n = 1$. From Lemma 3, we have $G = X\nabla Y$ with $|X|, |Y| \geq 1$. If both X and Y have no edge, then $G = K_{s,t}$ and thus $G \in \{K_2, K_{1,2}, K_{2,2}\}$ due to Lemma 15, where both $K_2 = (K_1 \cup K_0)\nabla K_1$ and $K_{1,2} = (K_1 \cup K_1)\nabla K_1$ have the form $(K_s \cup K_t)\nabla K_1$. Now we may assume that one of X and Y contains K_2, say X. Therefore, Lemma 16 implies that Y is $\{P_4, 3K_1, K_1 \cup K_{1,2}, 2K_2, K_1 \cup K_3, K_1\nabla(K_1 \cup K_2), K_{2,2}, K_2\nabla K_{1,2}\}$-free since G is P_4-free. Thus $Y \in \{2K_1, K_1 \cup K_2, K_{1,2}, K_1\nabla K_{1,2}, K_s\}$ due to Lemma 6.

If $Y = K_{1,2}$, then Lemma 16 implies that X is $\{2K_1, K_4\}$-free. Thus, Lemma 4 means that $X \in \{K_2, K_3\}$ since X contains K_2. Therefore, $G \in \{K_2\nabla K_{1,2}, K_3\nabla K_{1,2}\}$. If $Y = K_1 \cup K_2$, then Lemma 16 implies that X is $\{2K_2, K_3, 3K_1, K_{1,2}\}$-free. Thus Lemma 5 means that $X \in \{K_2, K_1 \cup K_2\}$ since X contains K_2. Note that, one may verify that $\lambda(M_{(K_1\cup K_2)\nabla(K_1\cup K_2)}) \leq -\sqrt{3}$ for any mixed graph with underlying graph $(K_1 \cup K_2)\nabla(K_1 \cup K_2)$ with the help of computer. Therefore, $G = K_2\nabla(K_1 \cup K_2)$. If $Y = K_s$ with $s \geq 2$, then Lemma 16 implies that X is $\{3K_1, K_1 \cup K_{1,2}, 2K_2, K_1 \cup K_3, P_4, K_1\nabla(K_1 \cup K_2), K_{2,2}, K_{1,3}, K_2\nabla K_{1,2}\}$-free since G is P_4-free. Thus, Lemma 6 means that $X \in \{K_1 \cup K_2, K_{1,2}, K_1\nabla K_{1,2}, K_r \mid r \geq 2\}$. It follows that $G \in \{(K_1 \cup K_2)\nabla K_s, K_{1,2}\nabla K_s, K_r\nabla K_s \mid s, r \geq 2\}$. Since G is $\{K_3\nabla(K_1\cup K_2), K_4\nabla K_{1,2}\}$-free, we have $G \in \{(K_1\cup K_2)\nabla K_2, K_{1,2}\nabla K_2, K_{1,2}\nabla K_3, K_n \mid n \geq 4\}$, where $K_n = (K_{n-1} \cup K_0)\nabla K_1$. If $Y = K_1\nabla K_{1,2}$, then Lemma 16 implies that X is $\{2K_1, K_3\}$-free. Thus, Lemma 4 means that $X = K_2$ since X contains K_2. Therefore, $G = K_3\nabla K_{1,2}$. If $Y = 2K_1$, then Lemma 16 implies that X is $\{3K_1, K_{1,2}, K_5\}$-free. Thus, Lemma 7 means that $X \in \{K_2, K_3, K_4, K_1 \cup K_2, K_1 \cup K_3, K_1 \cup K_4, 2K_2, K_2 \cup K_3, K_2 \cup K_4, 2K_3, K_3 \cup K_4, 2K_4\}$ since X contains K_2. Therefore, $G \in \{K_1\nabla K_{1,2}, K_2\nabla K_{1,2}, K_3\nabla K_{1,2}, (K_1 \cup K_2)\nabla 2K_1, (K_1 \cup$

$K_3)\nabla 2K_1, (K_1 \cup K_4)\nabla 2K_1, 2K_2\nabla 2K_1, (K_2 \cup K_3)\nabla 2K_1, (K_2 \cup K_4)\nabla 2K_1, 2K_3\nabla 2K_1, (K_3 \cup K_4)\nabla 2K_1, 2K_4\nabla 2K_1\}$.

In what follows, we consider the case of $Y = K_1$, that is $G = X\nabla K_1$. Since G is $K_{1,3}$-free according to Lemma 16, we have X is $3K_1$-free and X has at most two connected components. Suppose that X has two connected components, say X_1 and X_2 with $|X_1|, |X_2| \geq 1$. Then both X_1 and X_2 are $K_{1,2}$-free since otherwise X has an induced $3K_1$, and so X_1 and X_2 are complete graphs. Therefore, $G = (K_s \cup K_t)\nabla K_1$ with $s + t = n - 1$ and $s, t \geq 1$. Next we may assume that X is connected. Since X is P_4-free, we have $X = X_1\nabla Y_1$ with $|X_1|, |Y_1| \geq 1$. If both X_1 and Y_1 have no edge, then X is a bipartite graph and so $X \in \{K_2, K_{1,2}, K_{2,2}\}$ due to Lemma 15. Note that G is $K_1\nabla K_{2,2}$-free due to Lemma 16. Thus, $G \in \{K_3, K_{1,2}\nabla K_1\}$. Now we may assume X_1 contains K_2. Therefore, Lemma 16 implies that Y_1 is $\{P_4, 3K_1, K_1 \cup K_{1,2}, 2K_2, K_1 \cup K_3, K_1\nabla(K_1 \cup K_2), K_{2,2}, K_2\nabla K_{1,2}\}$-free since G is P_4-free. Thus $Y_1 \in \{2K_1, K_1 \cup K_2, K_{1,2}, K_1\nabla K_{1,2}, K_s\}$ due to Lemma 6. If $Y_1 = K_{1,2}$, then Lemma 16 implies that X_1 is $\{2K_1, K_3\}$-free. Thus, Lemma 4 means that $X_1 = K_2$ since X_1 contains K_2. Therefore, $G = K_3\nabla K_{1,2}$. If $Y_1 = K_1 \cup K_2$, then $G = X_1\nabla(K_1 \cup K_2)\nabla K_1$. Lemma 16 implies that X_1 is K_2-free, a contradiction. If $Y_1 = K_1\nabla K_{1,2}$, then $G = X_1\nabla(K_1\nabla K_{1,2})\nabla K_1 = X_1\nabla K_{1,2}\nabla K_2$. Lemma 16 implies that X_1 is K_2-free, a contradiction. If $Y_1 = 2K_1$, then $G = X_1\nabla 2K_1\nabla K_1 = X_1\nabla K_{1,2}$. Lemma 16 implies that X_1 is $\{2K_1, K_4\}$-free. Thus, Lemma 4 means that $X_1 \in \{K_2, K_3\}$ since X_1 contains K_2. Therefore, $G \in \{K_2\nabla K_{1,2}, K_3\nabla K_{1,2}\}$. If $Y_1 = K_s$, then $G = X_1\nabla K_{s+1}$. Lemma 16 implies that X_1 is $\{3K_1, K_1 \cup K_{1,2}, 2K_2, K_1 \cup K_3, P_4, K_1\nabla(K_1 \cup K_2), K_{2,2}, K_{1,3}, K_2\nabla K_{1,2}\}$-free since G is P_4-free. Thus, Lemma 6 means that $X_1 \in \{K_1 \cup K_2, K_{1,2}, K_1\nabla K_{1,2}, K_r \mid r \geq 2\}$. It follows that $G \in \{(K_1 \cup K_2)\nabla K_{s+1}, K_{1,2}\nabla K_{s+1}, K_r\nabla K_{s+1} \mid s \geq 1, r \geq 2\}$. Since G is $\{K_3\nabla(K_1 \cup K_2), K_4\nabla K_{1,2}\}$-free, we have $G \in \{(K_1 \cup K_2)\nabla K_2, K_{1,2}\nabla K_2, K_{1,2}\nabla K_3, K_n \mid n \geq 4\}$.

The proof is completed. □

For non-negative integers l, s, t, let $P_4(l, s, t)$ be the graph obtained from $K_l \cup K_s \cup (K_t\nabla P_4)$ by adding all edges between $V(K_l)$ and $V(K_t) \cup \{u_1, u_2\}$ and all edges between $V(K_s)$ and $\{u_2, u_3\}$, where $\{u_1, u_2, u_3, u_4\}$ is the vertex set of the P_4 with $u_i \sim u_{i+1}$ for $1 \leq i \leq 3$ (see Fig. 4). Clearly, $|V(P_4(l, s, t)| = 4 + l + s + t$ and $P_4(0, 0, 0) = P_4$. For non-negative integers x, y, z, let $K(x, y, z)$ be the graph obtained by identifying $v' \in V(K_x)$ and $v'' \in V(K_y)$, and then attaching z pendent vertices to distinct vertices of $V(K_y) \setminus \{v''\}$ (see Fig. 4). Clearly, $|V(K(x, y, z))| = x + y + z - 1$ and $K(1, n, 0) = K_n$.

Lemma 18. *If $M_G \in \mathcal{M}_n(-\sqrt{3})$ and G contains an induced P_4, then G belongs to*

$$\{P_4, P_4(1,0,0), P_4(2,0,0), P_4(3,0,0), P_4(0,0,1) = K_1\nabla P_4, P_4(0,0,2) = K_2\nabla P_4, P_4(2,1,0),$$
$$P_4(2,2,0), P_4(2,3,0), P_4(2,4,0), P_4(0,1,1), P_4(0,2,1), P_4(1,0,1), P_4(1,1,1),$$
$$G_{10}, G_{11}, G_{14}, G_{18}\} \cup \{K(l, s+2, k+1) \mid 2 \leq l \leq 3, s \geq 1, k \geq 0\} \cup \{K_s\nabla C_5 \mid s \geq 0\},$$

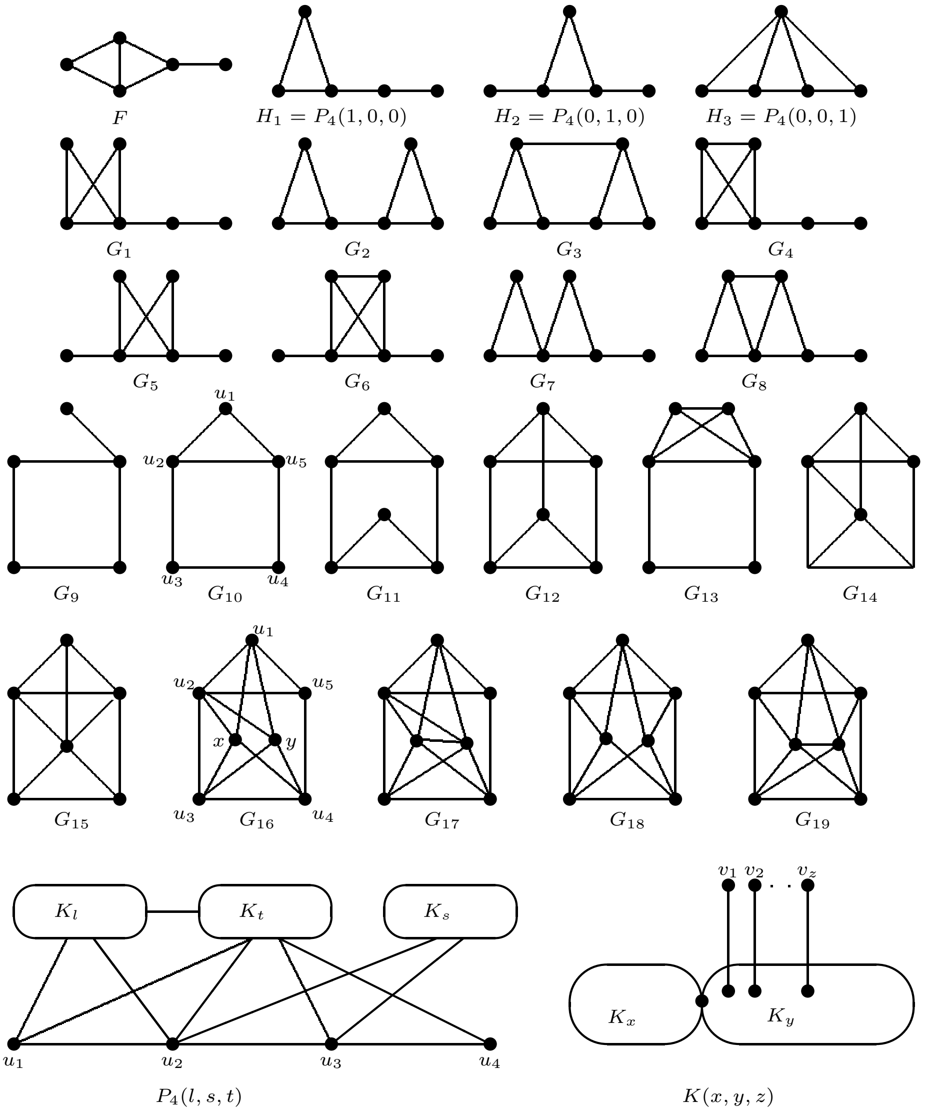

Fig. 4. The graphs used in Lemma 18.

where G_{10}, G_{11}, G_{14} *and* G_{18} *are shown in Fig. 4.*

Proof. Since G contains P_4, we have $n \geq 4$. If $n = 4$ then $G = P_4$. Now we assume $n \geq 5$. According to Lemma 16, G is $\mathcal{H}$-free where $\mathcal{H}$ is given in Lemma 16. This fact will be used frequently. In what follows, we divide into two cases to discuss.

Case 1. G contains induced C_5.

It is trivial that $G = C_5$ when $n = 5$. Now we assume that $n \geq 6$ and the vertex set of the induced C_5 is $U = \{u_1, u_2, u_3, u_4, u_5\}$ with $u_i \sim u_{i+1} \pmod 5$ for $1 \leq i \leq 5$. Denote by $X = \{x \in V(G) \setminus U \mid N_U(x) \neq \emptyset\}$ and $Y = V(G) \setminus (U \cup X)$. Since G is connected, we have $X \neq \emptyset$ and assume $x \in X$. If $|N_U(x)| = 1$, say $N_U(x) = \{u_1\}$, then $G[u_1, u_2, x, u_5] = K_{1,3}$, a contradiction. If $|N_U(x)| = 2$ then there are two types of structures, say $N_U(x) = \{u_1, u_2\}$ or $N_U(x) = \{u_1, u_3\}$. If the former case occurs then $G[x, u_2, u_3, u_4, u_5] = P_5$, a contradiction; if the latter case occurs, then $G[u_1, u_2, x, u_5] = K_{1,3}$, a contradiction. If $|N_U(x)| = 3$ then there are also two types of structures, say $N_U(x) = \{u_1, u_2, u_3\}$ or $N_U(x) = \{u_1, u_2, u_4\}$. If the former case occurs, then $G[x, u_1, u_2, u_3, u_4] = F$, a contradiction; if the latter case occurs, then $G[u_4, u_3, x, u_5] = K_{1,3}$, a contradiction. If $|N_U(x)| = 4$, say $N_U(x) = \{u_1, u_2, u_3, u_4\}$, then $G[x, u_1, u_2, u_3, u_5] = F$, a contradiction. Therefore, $N_U(x) = U$ for any $x \in X$. Moreover, for any $x_1, x_2 \in X$, we have $x_1 \sim x_2$. Otherwise, $G[x_1, u_1, x_2, u_3, u_2] = K_1 \nabla K_{2,2}$, a contradiction. Next we claim that $Y = \emptyset$. Otherwise, there exists $y \in Y$ such that $y \sim x$ for some $x \in X$ because G is connected. It leads to that $G[u_1, y, u_3, x] = K_{1,3}$, a contradiction. Hence $G = K_s \nabla C_5$ for $s \geq 0$.

Case 2. G is C_5-free.

Denote by $\mathcal{E}$ the set of connected induced subgraphs of G on 5 vertices. We will get the structure of G by investigating $\mathcal{E}$. We divide the following two subcases to discuss.

Subcase 2.1. For any graph $E \in \mathcal{E}$, E contains at most one of the induced P_4 or induced C_4.

Note that G contains induced P_4. Assume that the vertex set of the induced P_4 is $U = \{u_1, u_2, u_3, u_4\}$ with $u_i \sim u_{i+1}$ for $1 \leq i \leq 3$. Denote by $X = \{x \in V(G) \setminus U \mid N_U(x) \neq \emptyset\}$ and $Y = V(G) \setminus (U \cup X)$. Since G is connected, we have $X \neq \emptyset$. Clearly, for any $x \in X$, $G[U \cup \{x\}] \in \mathcal{E}$ contains induced P_4 and thus contains no induced C_4. Note that G is $\{P_5, K_{1,3}, F\}$-free. It leads to that $G[U \cup \{x\}]$ has three types of structures, denoted by $H_1 = P_4(1,0,0)$, $H_2 = P_4(0,1,0)$ and $H_3 = P_4(0,0,1)$ as shown in Fig. 4. Let $X_1 = \{x \in X \mid G[U \cup \{x\}] = H_1\}$, $X_2 = \{x \in X \mid G[U \cup \{x\}] = H_2\}$ and $X_3 = \{x \in X \mid G[U \cup \{x\}] = H_3\}$. Denote by $l = |X_1|$, $s = |X_2|$ and $t = |X_3|$.

For any $x_1, x_1' \in X_1$, the induced subgraph $G[U \cup \{x_1, x_1'\}]$ has four types of structures, denoted by G_i for $1 \leq i \leq 4$ as shown in Fig. 4. Note that G_1 contains an induced $K_{1,3}$, $\lambda(M_{G_2}) = \lambda(G_2) = -\sqrt{3}$ and G_3 contains an induced subgraph on 5 vertices which contains both induced P_4 and induced C_4. Therefore, $G[U \cup \{x_1, x_1'\}] = G_4$. It means that $G[X_1] = K_l$ and $G[U \cup X_1] = K_{l+2} \cdot K_2$ is the graph obtained by adding an edge between the complete graph K_{l+2} and K_2. Moreover, by Lemma 13, we have $\lambda(M_G[U \cup X_1]) = \lambda(K_{l+2} \cdot K_2) \leq \lambda(K_6 \cdot K_2) = -1.735 < -\sqrt{3}$ when $l \geq 4$. Thus, we have $l \leq 3$.

For any $x_2, x_2' \in X_2$, the induced subgraph $G[U \cup \{x_2, x_2'\}]$ has two types of structures, denoted by G_5 and G_6 as shown in Fig. 4. Note that G_5 contains induced $K_{1,3}$. Therefore, $G[U \cup \{x_2, x_2'\}] = G_6$. It means that $G[X_2] = K_s$.

For any $x_3, x_3' \in X_3$, we have $G[U \cup \{x_3, x_3'\}] \in \{2K_1 \nabla P_4, K_2 \nabla P_4\}$ and thus $G[U \cup \{x_3, x_3'\}] = K_2 \nabla P_4$ since G is $2K_1 \nabla P_3$-free. It means that $G[X_3] = K_t$ and $G[U \cup X_3] = K_t \nabla P_4$. Note that G is $K_3 \nabla (K_1 \cup K_2)$-free. We have G is $K_3 \nabla P_4$-free, and thus $t \leq 2$.

For any $x_1 \in X_1$ and $x_2 \in X_2$, we have $G[U \cup \{x_1, x_2\}] \in \{G_7, G_8\}$ as shown in Fig. 4. Note that G_8 contains induced F. Therefore, $G[U \cup \{x_1, x_2\}] = G_7$. It means that there

Table 1
The smallest eigenvalues of graphs in $\mathcal{P}$.

P:	$P_4(0,1,2)$	$P_4(0,3,1)$	$P_4(2,0,1)$	$P_4(1,0,2)$	$P_4(2,5,0)$	$P_4(3,1,0)$	$P_4(1,2,1)$
$\lambda(P)$:	-1.85	-1.76	-1.76	-1.84	-1.733	$-\sqrt{3}$	$-\sqrt{3}$

is no edge between X_1 and X_2. Similarly, each vertex in X_1 is adjacent to every vertex of X_3 and there is no edge between X_2 and X_3.

By the arguments above, the induced subgraph $G[U \cup X] = P_4(l,s,t)$ with $l \leq 3$ and $t \leq 2$. We will further analyze the triple (l,s,t) by investigating some more forbidden subgraphs of G.

Denote by $\mathcal{P} = \{P_4(0,1,2), P_4(0,3,1), P_4(2,0,1), P_4(1,0,2), P_4(2,5,0), P_4(3,1,0), P_4(1,2,1)\}$. For any $P \in \mathcal{P}$, if P is an induced subgraph of G, then M_P is an induced mixed subgraph of M_G, and thus $\lambda(M_P) > -\sqrt{3}$. Therefore, each mixed triangle in M_P belongs to $\mathcal{C}_3$. Lemma 13 indicates that $\lambda(M_P) = \lambda(P)$. However, by immediate calculations, $\lambda(P) \leq -\sqrt{3}$ for any $P \in \mathcal{P}$ (see Table 1). It means that G is $\mathcal{P}$-free. Since $X \neq \emptyset$, we have $l+s+t > 0$. Suppose that there are exactly two of l,s,t being 0. If $l > 0$ then $(l,s,t) \in \{(1,0,0),(2,0,0),(3,0,0)\}$. If $s > 0$ then $(l,s,t) \in \{(0,s,0) \mid s \geq 1\}$. If $t > 0$ then $(l,s,t) \in \{(0,0,1),(0,0,2)\}$. Suppose that there is exactly one of l,s,t being 0. If $l = 0$ then we have $(l,s,t) \in \{(0,1,1),(0,2,1)\}$ because G is $\{P_4(0,1,2), P_4(0,3,1)\}$-free. If $s = 0$ then we have $(l,s,t) = (1,0,1)$ because G is $\{P_4(2,0,1), P_4(1,0,2)\}$-free. If $t = 0$ then we have $(l,s,t) \in \{(1,s,0) \mid s \geq 1\} \cup \{(2,s,0) \mid 1 \leq s \leq 4\}$ because G is $\{P_4(2,5,0), P_4(3,1,0)\}$-free. Suppose that none of l,s,t is 0. Then we have $(l,s,t) = (1,1,1)$ because G is $\{P_4(1,2,1), P_4(2,0,1), P_4(1,0,2)\}$-free.

Next, we consider Y. If $Y = \emptyset$ then $G = G[U \cup X]$ belongs to

$$\{P_4(1,0,0), P_4(2,0,0), P_4(3,0,0), P_4(0,0,1), P_4(0,0,2), P_4(0,1,1), P_4(0,2,1), P_4(1,0,1), P_4(1,1,1)\} \cup \{P_4(1,s,0) \mid s \geq 1\} \cup \{P_4(0,s,0) \mid s \geq 1\} \cup \{P_4(2,s,0) \mid 1 \leq s \leq 4\}.$$

In what follows, we assume $Y \neq \emptyset$ and $|Y| = k \geq 1$.

Since G is connected, there exists $y \in Y$ such that $N_X(y) \neq \emptyset$. If $y \sim x_1$ for some $x_1 \in X_1$ then $G[y, x_1, u_2, u_3, u_4] = P_5$, a contradiction; if $y \sim x_3$ for some $x_3 \in X_3$ then $G[y, x_3, u_1, u_4] = K_{1,3}$, a contradiction. It means that $N_X(Y) \subseteq X_2$. Moreover, for some $y \in Y$, if $N_{X_2}(y) = \emptyset$ then the distance $d_G(y, x_2) \geq 2$ for any $x_2 \in X_2$. It leads to that $d_G(y, u_4) \geq 4$, a contradiction. It yields that, for any $y \in Y$, we have $y \sim x_2$ for some $x_2 \in X_2$, and thus $s = |X_2| \geq 1$. Besides, for any $y_1, y_2 \in N_Y(x_2)$, we have $y_1 \sim y_2$ since otherwise $G[y_1, y_2, x_2, u_2] = K_{1,3}$, a contradiction. It means that $N_Y(x_2)$ is a clique for any $x_2 \in X_2$. Moreover, for any $y \in Y$, we have $|N_{X_2}(y)| = 1$ otherwise there exist $x_2, x_2' \in N_{X_2}(y)$ and thus $G[y, x_2, x_2', u_2, u_1] = F$, a contradiction. Now we claim that $t = 0$. Otherwise, $G[y, x_2, u_2, u_3, x_3] = F$ for some $y \in Y$, $x_2 \in X_2$ with $y \sim x_2$ and $x_3 \in X_3$, which is impossible. Note that $l \leq 2$ when $s \geq 1$ by the form of $G[U \cup X] = P_4(l,s,t)$. If $l = 2$ then $\lambda(M_{G[X_1 \cup U \cup \{x_2, y\}]}) = \lambda(G[X_1 \cup U \cup \{x_2, y\}]) = -\sqrt{3}$ for some $y \in Y$, $x_2 \in X_2$ and $y \sim x_2$, which is impossible. Therefore, we have $l \leq 1$. If

$l = 1$, then $|N_Y(x_2)| \leq 1$ for any $x_2 \in X_2$. Otherwise, there exist $y_1, y_2 \in N_Y(x_2)$ for some $x_2 \in X_2$, and thus $G[u_1, x_1, u_2, x_2, y_1, y_2] = G_2$, which is impossible. Therefore, G is obtained from $P_4(1, s, 0)$ by attaching k pendent vertices to distinct vertices of X_2, that is, $G = K(3, s+2, k+1)$ with $s, k \geq 1$. Now assume that $l = 0$. If there exists $x_2 \in X_2$ such that $|N_Y(x_2)| \geq 2$ then $G[N_Y(x_2) \cup X_2 \cup \{u_1, u_2, u_3\}] = P_4(|N_Y(x_2) - 1|, s, 0) = P_4(l', s, 0)$ with $l' \geq 1$, which has been discussed before. Therefore, we may assume $|N_Y(x_2)| \leq 1$ for any $x_2 \in X_2$. Thus, G is obtained from $P_4(0, s, 0)$ by attaching k pendent vertices to distinct vertices of X_2, that is, $G = K(2, s + 2, k + 1)$ with $s, k \geq 1$.

Note that $P_4(0, s, 0) = K(2, s+2, 1)$ and $P_4(1, s, 0) = K(3, s+2, 1)$. Thus, we conclude that G belongs to

$$\{P_4(1,0,0), P_4(2,0,0), P_4(3,0,0), P_4(0,0,1), P_4(0,0,2), P_4(0,1,1), P_4(0,2,1), P_4(1,0,1), \\ P_4(1,1,1)\} \cup \{K(l, s+2, k+1) \mid 2 \leq l \leq 3, s \geq 1, k \geq 0\} \cup \{P_4(2, s, 0) \mid 1 \leq s \leq 4\}.$$

Subcase 2.2. There exists $E \in \mathcal{E}$ such that E contains both induced P_4 and induced C_4.

Let the graphs $G_9, G_{10}, \ldots, G_{19}$ be the graphs shown in Fig. 4. To prove this subcase, we need the following claims.

Claim 1. *G contains an induced G_{10}.*

Proof of Claim 1. Since E contains both induced P_4 and induced C_4, we have $E \in \{G_9, G_{10}\}$. It is clear that G_9 is not an induced subgraph of G because G_9 contains an induced $K_{1,3}$. Thus, $E = G_{10}$ is an induced G_{10}. □

Claim 2. *If H is a connected induced subgraph of G on 6 vertices containing G_{10} then $H \in \{G_{11}, G_{14}\}$.*

Proof of Claim 2. Assume that $V(H) = \{u_1, u_2, u_3, u_4, u_5, x\}$ and $\{u_1, u_2, u_3, u_4, u_5\}$ induces the subgraph $E = G_{10}$ as shown in Fig. 4. Keep in mind that $G[u_1, u_2, u_3, u_4] = G[u_1, u_5, u_4, u_3] = P_4$. Now we consider the induced subgraphs $G[u_1, u_2, u_3, u_4, x]$, $G[u_1, u_5, u_4, u_3, x]$. If they do not contain induced C_4, then we have

$$G[u_1, u_2, u_3, u_4, x], G[u_1, u_5, u_4, u_3, x] \in \{P_4(1,0,0), P_4(0,1,0), P_4(0,0,1)\}$$

according to the argument in Subcase 2.1. If they contain induced C_4, then each of them is either G_9 or G_{10}. Furthermore, since G contains no induced G_9, we have both of them are G_{10}. It yields that $G[u_1, u_2, u_3, u_4, x], G[u_1, u_5, u_4, u_3, x] \in \{P_4(1,0,0), P_4(0,1,0), P_4(0,0,1), G_{10}\}$, and thus $H \in \{G_{11}, G_{12}, G_{13}, G_{14}, G_{15}\}$. However, one may check that $\lambda(M_{H'}) \leq -\sqrt{3}$ whenever $H' \in \{G_{12}, G_{13}\}$ with the help of computer,[1] and G_{15} contains an induced $K_1 \nabla K_{2,2}$. Thus, $H \in \{G_{11}, G_{14}\}$. □

[1] The graph H' contains 9 edges, and thus there are at most 3^9 types of $M_{H'}$. The spectra of such mixed graphs could be obtained just by, for example, *Matlab*.

Claim 3. *If H is a connected induced subgraph of G on 7 vertices containing G_{10} then $H = G_{18}$.*

Proof of Claim 3. According Claim 2, we may assume that $V(H) = \{u_1, u_2, u_3, u_4, u_5, x, y\}$ such that $\{u_1, u_2, u_3, u_4, u_5\}$ induces the subgraph $E = G_{10}$ and $\{u_1, u_2, u_3, u_4, u_5, x\}$ induces the subgraph $H_0 \in \{G_{11}, G_{14}\}$. We first assume that $H_0 = G_{11}$. We conclude that $N_{V(E)}(y) \neq \emptyset$. Otherwise, $y \sim x$, and thereby Claim 2 indicates that $G[u_2, u_3, u_4, u_5, x, y] \in \{G_{11}, G_{14}\}$ because $G[u_2, u_3, u_4, u_5, x] = G_{10}$. It is impossible as y is a pendent vertex in $G[u_2, u_3, u_4, u_5, x, y]$. According to Claim 2 again, we have $G[V(E) \cup \{y\}] \in \{G_{11}, G_{14}\}$ due to $N_{V(E)}(y) \neq \emptyset$. It leads to that $G[u_2, u_3, u_4, u_5, x, y] \notin \{G_{11}, G_{14}\}$. By noticing that $G[u_2, u_3, u_4, u_5, x] = G_{10}$ and $N_{G[u_2,u_3,u_4,u_5,x]}(y) \neq \emptyset$, Claim 2 indicates that $G[u_2, u_3, u_4, u_5, x, y] \in \{G_{11}, G_{14}\}$, a contradiction. Next, we assume that $H_0 = G_{14}$. Similarly, we conclude that $N_{V(E)}(y) \neq \emptyset$, and thus $G[V(E) \cup \{y\}] \in \{G_{11}, G_{14}\}$. If $G[V(E) \cup \{y\}] = G_{11}$ then $G[u_2, x, u_4, u_5, y] = G_9$ when $x \nsim y$, and $G[u_2, u_3, x, u_5, y] = F$ when $x \sim y$, which are both impossible. If $G[V(E) \cup \{y\}] = G_{14}$ then $G[V(E) \cup \{x, y\}]$ has four types of structures, denoted by G_{16}, G_{17}, G_{18} and G_{19} as shown in Fig. 4. Note that $G[u_1, x, u_3, u_4, u_5] = G_{10}$ and $N_{G[u_1,x,u_3,u_4,u_5]}(y) \neq \emptyset$. We have $G[u_1, x, u_3, u_4, u_5, y] \in \{G_{11}, G_{14}\}$ by Claim 2, which only occurs in the case of $G[V(E) \cup \{x, y\}] = G_{18}$. Hence $H = G[V(E) \cup \{x, y\}] = G_{18}$. $\square$

Now we consider the structure of G. If $n = 5$ then Claim 1 means that $G = G_{10}$. If $n = 6$ then $G \in \{G_{11}, G_{14}\}$ according to Claims 1 and 2. If $n = 7$ then $G = G_{18}$ according to Claims 1 and 3. If $n \geq 8$ then Claims 1 and 3 indicates that there exist vertices $u_1, u_2, \ldots, u_5, x, y, z$ such that $\{u_1, u_2, \ldots, u_5\}$ induces $E = G_{10}$, $\{u_1, u_2, \ldots, u_5, x, y\}$ induces $H = G_{18}$ and $N_{V(H)}(z) \neq \emptyset$. By the symmetry of the vertices x and y in H, we may assume that $N_{V(E)\cup\{x\}}(z) \neq \emptyset$. Therefore, Claim 3 implies that $G[u_1, u_2, \ldots, u_5, x, z] = G_{18}$, and thereby $G[u_1, u_2, \ldots, u_5, y, z] \neq G_{18}$, a contradiction.

The proof is completed. $\square$

Combining Lemmas 17 and 18, one of our main result is obtained.

Theorem 1. *If $M_G \in \mathcal{M}_n(-\sqrt{3})$ then G belongs to $\mathcal{G}_0 \cup \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$, where*

$$\begin{cases} \mathcal{G}_0 = \{K_{2,2}, K_1\nabla K_{1,2}, K_2\nabla K_{1,2}, K_3\nabla K_{1,2}, K_2\nabla(K_1 \cup K_2), 2K_2\nabla 2K_1, 2K_3\nabla 2K_1, \\ \quad 2K_4\nabla 2K_1, (K_2 \cup K_1)\nabla 2K_1, (K_1 \cup K_3)\nabla 2K_1, (K_1 \cup K_4)\nabla 2K_1, (K_2 \cup K_3)\nabla 2K_1, \\ \quad (K_2 \cup K_4)\nabla 2K_1, (K_3 \cup K_4)\nabla 2K_1, P_4, P_4(1,0,0), P_4(2,0,0), P_4(3,0,0), \\ \quad P_4(0,0,1) = K_1\nabla P_4, P_4(0,0,2) = K_2\nabla P_4, P_4(2,1,0), P_4(2,2,0), P_4(2,3,0), \\ \quad P_4(2,4,0), P_4(0,1,1), P_4(0,2,1), P_4(1,0,1), P_4(1,1,1), G_{10}, G_{11}, G_{14}, G_{18}\}, \\ \mathcal{G}_1 = \{(K_s \cup K_t)\nabla K_1 \mid s,t \geq 0, s+t = n-1\}, \\ \mathcal{G}_2 = \{K_s\nabla C_5 \mid s \geq 0\}, \\ \mathcal{G}_3 = \{K(l, s+2, k+1) \mid 2 \leq l \leq 3, s \geq 1, k \geq 0\}. \end{cases}$$

In what follows, we determine M_G by considering its underlying graphs as given in Theorem 1.

Theorem 2. *Let M_G be a mixed graph with underlying graph $G = (K_s \cup K_t)\nabla K_1$. Then, $\lambda(M_G) > -\sqrt{3}$ if and only if $M_G \in \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3 \cup \mathcal{M}_4$, where*

$$\begin{cases} \mathcal{M}_1 = [K_n] = \{K_n[s,t] \mid s,t \geq 0, s+t=n\}, \\ \mathcal{M}_2 = [2K_4\nabla K_1] \cup [(K_5 \cup K_4)\nabla K_1], \\ \mathcal{M}_3 = \bigcup\limits_{3\leq s\leq 9} [(K_s \cup K_3)\nabla K_1], \\ \mathcal{M}_4 = [(K_s \cup K_2)\nabla K_1] \cup [(K_s \cup K_1)\nabla K_1]. \end{cases}$$

Proof. We first consider the spectrum of G. Assume that π: $V(G) = V_1 \cup \{v\} \cup V_2$ is the partition such that $G[V_1 \cup \{v\}] = K_{s+1}$ and $G[V_2 \cup \{v\}] = K_{t+1}$. The Hermitian matrix of G is

$$H(G) = \begin{pmatrix} J_s - I_s & \mathbf{1}_s & \mathbf{0}_{s\times t} \\ \mathbf{1}_s^T & 0 & \mathbf{1}_t^T \\ \mathbf{0}_{t\times s} & \mathbf{1}_t & J_t - I_t \end{pmatrix},$$

where J, I, $\mathbf{1}$ and $\mathbf{0}$ are respectively the all-one matrix, identity matrix, all-one vector and zero matrix with the corresponding size. Therefore, Lemma 2 indicates that π is an equitable partition with quotient matrix

$$H_\pi = \begin{pmatrix} s-1 & 1 & 0 \\ s & 0 & t \\ 0 & 1 & t-1 \end{pmatrix}.$$

Assume that $V_1 = \{v_1, v_2, \ldots, v_s\}$ and $V_2 = \{u_1, u_2, \ldots, u_t\}$. For $2 \leq j \leq s$ and $2 \leq k \leq t$, let $\delta_{1,j} \in \mathbb{R}^s$ be the vector indexed by V_1 such that $\delta_{1,j}(v_1) = 1$, $\delta_{1,j}(v_j) = -1$ and $\delta_{1,j}(v_{j'}) = 0$ for $j' \notin \{1,j\}$ and let $\delta_{2,k} \in \mathbb{R}^t$ be the vector indexed by V_2 such that $\delta_{2,k}(u_1) = 1$, $\delta_{2,k}(u_k) = -1$ and $\delta_{2,k}(u_{k'}) = 0$ for $k' \notin \{1,k\}$. It is easy to see that $H(G)(\delta_{1,j}^T\ \mathbf{0})^T = -(\delta_{1,j}^T\ \mathbf{0})^T$ and $H(G)(\mathbf{0}\ \delta_{2,k}^T)^T = -(\mathbf{0}\ \delta_{2,k}^T)^T$ for any j and k, and thus $H(G)$ has an eigenvalue -1 with multiplicity at least $s+t-2 = n-3$. Lemma 2 implies that the other three eigenvalues of G are just the roots $\epsilon_1 \geq \epsilon_2 \geq \epsilon_3$ of the function $f_{s,t}(x) = det(xI - H_\pi) = x^3 + (2-t-s)x^2 + (st-2t-2s+1)x - s - t + 2st$. It is clear that $f_{s,t}(0) = 2st - s - t \geq 0$.

Now we show the necessity. If $s = 0$ or $t = 0$ then $G = K_n$. Since $\lambda(M_G) > -\sqrt{3}$, Lemma 11 means that any mixed triangle of M_G belongs to $\mathcal{C}_3$. Hence, Lemma 14 indicates that $M_G = M_{K_n} \in \{K_n[s,t] \mid s,t \geq 0, s+t=n\} = [K_n] = \mathcal{M}_1$. Suppose $G = (K_s \cup K_t)\nabla K_1$ with $s,t \geq 1$ and $s \geq t$. Note that G contains no induced cycle with length greater than 3, Lemma 13 indicates that $M_G \in [G]$ and thus $\mathrm{Sp}(M_G) = \mathrm{Sp}(G)$. Therefore, the eigenvalues of M_G are the eigenvalues of $H(G)$. Note that $\epsilon_1 > 0$. By the images of the function $f_{s,t}(x)$, $\epsilon_3 > -\sqrt{3}$ if and only if $f_{s,t}(-\sqrt{3}) < 0$. If $t \geq 5$ then

$$f_{s,t}(-\sqrt{3}) = (2-\sqrt{3})(st-2s-2t)+6-4\sqrt{3} \geq (2-\sqrt{3})t+6-4\sqrt{3} \geq 16-9\sqrt{3} \geq 0,$$

and thus $\epsilon_3 \leq -\sqrt{3}$. Therefore, we may assume $t \leq 4$. Note that $f_{s,4}(-\sqrt{3}) = (4-2\sqrt{3})s+4\sqrt{3}-10 < 0$ if and only if $4 \leq s \leq 5$, $f_{s,3}(-\sqrt{3}) = (2-\sqrt{3})s+2\sqrt{3}-6 < 0$ if and only if $3 \leq s \leq 9$, $f_{s,2}(-\sqrt{3}) = -2 < 0$ for any $s \geq 2$, and $f_{s,1}(-\sqrt{3}) = (\sqrt{3}-2)s-2-2\sqrt{3} < 0$ for any $s \geq 1$. It leads to that $M_G \in \mathcal{M}_2 \cup \mathcal{M}_3 \cup \mathcal{M}_4$.

Next, we show the sufficiency. For any $M_G \in \mathcal{M}_1$, Lemma 14 implies that $\lambda(M_G) = -1$. For any $M_G \in \mathcal{M}_2$, its smallest eigenvalue is the smallest root of $f_{s,4}(x) = x^3 - (2+s)x^3 + (2s-7)x + 7s - 4$ for $4 \leq s \leq 5$. Note that $f_{s,4}(0) = 7s-4 > 0$ and $f_{s,4}(-\sqrt{3}) = (4-2\sqrt{3})s+4\sqrt{3}-10 < 0$. The smallest root of $f_{s,4}(x)$ is greater than $-\sqrt{3}$ by the image of $f_{s,4}(x)$, and thus $\lambda(M_G) > -\sqrt{3}$. For any $M_G \in \mathcal{M}_3$, it smallest eigenvalue is the smallest root of $f_{s,3}(x) = x^3-(1+s)x^2+(s-5)x+5s-3$ for $3 \leq s \leq 9$. Note that $f_{s,3}(0) = 5s-3 > 0$ and $f_{s,3}(-\sqrt{3}) = (2-\sqrt{3})s+2\sqrt{3}-6 < 0$. The smallest root of $f_{s,3}(x)$ is greater than $-\sqrt{3}$ by the image of $f_{s,3}(x)$, and thus $\lambda(M_G) > -\sqrt{3}$. For $M_G \in \mathcal{M}_4$, if $M_G \in [(K_s \cup K_2)\nabla K_1]$, its smallest eigenvalue is the smallest root of $f_{s,2}(x) = x^3 - sx^2 - 3x + 3s - 2$ for $s \geq 2$; if $M_G \in [(K_s \cup K_1)\nabla K_1]$, its smallest eigenvalue is the smallest root of $f_{s,1}(x) = x^3-(s-1)x^2-(s+1)x+s-1$ for $s \geq 1$. Note that $f_{s,2}(0) = 3s-2, f_{s,1}(0) = s-1 \geq 0$, and $f_{s,2}(-\sqrt{3}) = -2, f_{s,1}(-\sqrt{3}) = (\sqrt{3}-2)s-2-2\sqrt{3} < 0$. The smallest roots of $f_{s,2}(x)$ and $f_{s,1}(x)$ are greater than $-\sqrt{3}$ by the image of them, and thus $\lambda(M_G) > -\sqrt{3}$.

The proof is completed. $\square$

Theorem 3. *Let M_G be a mixed graph with $G \in \{K(l, s+2, k+1)$ for $2 \leq l \leq 3$, $s \geq 1$ and $k \geq 0$. Then $\lambda(M_G) > -\sqrt{3}$ if and only if $M_G \in [G]$.*

Proof. Firstly, we show the necessity. Since $\lambda(M_G) > -\sqrt{3}$, Lemma 11 means that any mixed triangle of M_G belongs to $\mathcal{C}_3$. Note that G contains no induced cycle with length greater than 3, Lemma 13 indicates that $M_G \in [G]$. In what follows, we show the sufficiency. Note that $G = K(2, s+2, k+1)$ could be obtained by attaching $k+2$ pendent vertices to distinct vertices of K_{s+2}. Let $\pi : V(G) = V_1 \cup V_2 \cup V_3$ be the partition of $V(G)$, where V_1 is the set of vertices with degree $s+2$, V_2 is the set of vertices with degree $s+1$ and V_3 is the set of pendent vertices. Clearly, $|V_1| = |V_3| = k+2$ and $|V_2| = s-k$. The Hermitian matrix of G is

$$H(G) = \begin{pmatrix} J_{k+2} - I_{k+2} & J_{(k+2)\times(s-k)} & I_{k+2} \\ J_{(s-k)\times(k+2)} & J_{s-k} - I_{s-k} & \mathbf{0}_{(s-k)\times(k+2)} \\ I_{k+2} & \mathbf{0}_{(k+2)\times(s-k)} & \mathbf{0}_{(k+2)\times(k+2)} \end{pmatrix},$$

where J, I and $\mathbf{0}$ are respectively the all-one matrix, identity matrix and zero matrix with the corresponding size. Therefore, Lemma 2 indicates that π is an equitable partition with quotient matrix

$$H_\pi = \begin{pmatrix} k+1 & s-k & 1 \\ k+2 & s-k-1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Assume that $V_2 = \{u_1, u_2, \ldots, u_{s-k}\}$. For $2 \le b \le s-k$, let $\delta_{2,b}$ be the vector indexed by V_2 such that $\delta_{2,b}(u_1) = 1$, $\delta_{2,b}(u_b) = -1$ and $\delta_{2,b}(u_{b'}) = 0$ for $b' \notin \{1, b\}$. It is easy to see that $H(G)(\mathbf{0}\ \delta_{2,b}^T\ \mathbf{0})^T = -(\mathbf{0}\ \delta_{2,b}^T\ \mathbf{0})^T$ for any $2 \le b \le s-k$. It means that -1 is an eigenvalue of $H(G)$ with multiplicity at least $s-k-1$.

Assume that $V_1 = \{v_1, v_2, \ldots, v_{k+2}\}$ and $V_3 = \{w_1, w_2, \ldots, w_{k+2}\}$. For $2 \le a, c \le k+2$, let $\delta_{1,a}$ be the vector indexed by V_1 such that $\delta_{1,a}(v_1) = \frac{-1-\sqrt{5}}{2}$, $\delta_{1,a}(v_a) = \frac{1+\sqrt{5}}{2}$ and $\delta_{1,a}(v_{a'}) = 0$ for $a' \notin \{1, a\}$; let $\delta_{3,c}$ be the vector indexed by V_3 such that $\delta_{3,c}(w_1) = 1$, $\delta_{3,c}(w_c) = -1$ and $\delta_{3,c}(w_{c'}) = 0$ for $c' \notin \{1, c\}$. It is easy to verify that, for any $2 \le a \le k+2$,

$$H(G)(\delta_{1,a}^T\ \mathbf{0}\ \delta_{3,a}^T)^T = \frac{-1-\sqrt{5}}{2}(\delta_{1,a}^T\ \mathbf{0}\ \delta_{3,a}^T)^T.$$

It yields that $\frac{-1-\sqrt{5}}{2}$ is an eigenvalue of $H(G)$ with multiplicity at least $k+1$.

For $2 \le a \le k+2$, let $\eta_{1,a}(v_1) = \frac{-1+\sqrt{5}}{2}$, $\eta_{1,a}(v_a) = \frac{1-\sqrt{5}}{2}$ and $\eta_{1,a}(v_{a'}) = 0$ for $a' \notin \{1, a\}$. It is easy to verify that

$$H(G)(\eta_{1,a}^T\ \mathbf{0}\ \delta_{3,a}^T)^T = \frac{-1+\sqrt{5}}{2}(\eta_{1,a}^T\ \mathbf{0}\ \delta_{3,a}^T)^T.$$

It indicates that $\frac{-1+\sqrt{5}}{2}$ is an eigenvalue of $H(G)$ with multiplicity at least $k+1$.

Now Lemma 2 implies that the other three eigenvalues of G are just the roots $\epsilon_1 \ge \epsilon_2 \ge \epsilon_3$ of the function $f(x) = det(xI - H_\pi) = x^3 - sx^2 - (s+2)x + s - k - 1$. Note that $\epsilon_1 > 0$, $f(-1) = s - k \ge 0$ and $f(-\sqrt{3}) = -(2-\sqrt{3})s - k - \sqrt{3} - 1 < 0$. Therefore, $\epsilon_3 > -\sqrt{3}$ for any $s \ge 1$ and $k \ge 0$, that is $\lambda(M_G) = \lambda(G) > -\sqrt{3}$. Similarly, for $G = K(3, s+2, k+1)$, we have $\lambda(M_G) > -\sqrt{3}$ for any $s \ge 1$ and $k \ge 0$. □

Theorem 4. *Let M_G be a mixed graph with underlying graph $G = K_s \nabla C_5$. If each mixed triangle of M_G belongs to $\mathcal{C}_3$ and each mixed induced pentagon of M_G belongs to $\mathcal{C}_5$ then $M_G \in [G]$. Furthermore, $\lambda(M_G) > -\sqrt{3}$ if and only if $M_G \in [K_s \nabla C_5]$ with $s \le 2$.*

Proof. If each mixed triangle of M_G belongs to $\mathcal{C}_3$ and each mixed induced pentagon of M_G belongs to $\mathcal{C}_5$, then M_G is positive and thus $M_G \in [G]$ by Lemma 9.

Let $\pi : V(K_s) \cup V(C_5)$ be a partition of $V(G)$. The Hermitian matrix of G is

$$H(G) = \begin{pmatrix} J_s - I_s & J_{s\times 5} \\ J_{5\times s} & H(C_5) \end{pmatrix},$$

where J and I are respectively the all-one matrix and identity matrix with the corresponding size. Therefore, Lemma 2 indicates that π is an equitable partition with quotient matrix

$$H_\pi = \begin{pmatrix} s-1 & 5 \\ s & 2 \end{pmatrix}.$$

Assume that $V(K_s) = \{v_1, v_2, \ldots, v_s\}$. For $1 < j \leq s$, let $\delta_j \in \mathbb{R}^s$ be the vector indexed by $V(K_s)$ such that $\delta_j(v_1) = 1$, $\delta_j(v_j) = -1$ and $\delta_j(v_{j'}) = 0$ for $j' \notin \{1, j\}$. It is easy to see that $H(G)(\delta_j^T\ \mathbf{0})^T = -(\delta_j^T\ \mathbf{0})^T$ for any j. It's well known that the vectors $\theta_k = (1\ \omega^k\ \omega^{2k}\ \omega^{3k}\ \omega^{4k})^T$ for $1 \leq k \leq 4$ are eigenvectors of $H(C_5)$ where $\omega = e^{\frac{2\pi i}{5}}$ and the corresponding eigenvalues are $\mu_k = \omega^k + \omega^{4k} = 2\cos(2k\pi/5) > -\sqrt{3}$. Note that $\mathbf{1}^T\theta_k = 0$. We have $H(G)(\mathbf{0}\ \theta_i^T)^T = \mu_i(\mathbf{0}\ \theta_i^T)^T$. Therefore, H has eigenvalues -1 with multiplicity at least $s-1$ and μ_i for $1 \leq i \leq 4$. Lemma 2 implies that the other 2 eigenvalues of G are just the roots $\epsilon_1 \geq \epsilon_2$ of the function $f(x) = det(xI - H_\pi) = x^2 - (s+1)x - (3s+2)$. Note that $\epsilon_1 > 0$. Since $f(-\sqrt{3}) = -(3-\sqrt{3})s + \sqrt{3} + 1 > 0$ if and only if $s < \frac{1+\sqrt{3}}{3-\sqrt{3}} \approx 2.1547$, we have $\epsilon_2 > -\sqrt{3}$ if and only if $s \leq 2$. $\square$

Now we are ready to present our main result.

Theorem 5. *Let M_G be a connected mixed graph on n vertices with $n \geq 11$. Then $\lambda(M_G) > -\sqrt{3}$ if and only if $M_G \in \mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{H}_3 \cup \mathcal{H}_4 \cup \mathcal{H}_5 \cup \mathcal{H}_6$, where*

$$\begin{cases} \mathcal{H}_1 & = & \bigcup\limits_{n\geq 11} [K_n], \\ \mathcal{H}_2 & = & [(K_7 \cup K_3)\nabla K_1] \cup [(K_8 \cup K_3)\nabla K_1] \cup [(K_9 \cup K_3)\nabla K_1], \\ \mathcal{H}_3 & = & \bigcup\limits_{s\geq 8} [(K_s \cup K_2)\nabla K_1], \\ \mathcal{H}_4 & = & \bigcup\limits_{s\geq 9} [(K_s \cup K_1)\nabla K_1], \\ \mathcal{H}_5 & = & \bigcup\limits_{s+k\geq 7} [K(2, s+2, k+1)], \\ \mathcal{H}_6 & = & \bigcup\limits_{s+k\geq 6} [K(3, s+2, k+1)]. \end{cases}$$

Proof. For $M_G \in \cup_{1\leq i\leq 4}\mathcal{H}_i$, Theorem 2 implies that $\lambda(M_G) > -\sqrt{3}$. From Theorem 3, each graph in $\mathcal{H}_5 \cup \mathcal{H}_6$ has smallest eigenvalue greater than $-\sqrt{3}$. The sufficiency holds.

In what follows, we show the necessity. From Theorem 1, the underlying graph G with $|V(G)| \geq 11$ belongs to $\{(K_s \cup K_t)\nabla K_1 \mid s, t \geq 0, s+t \geq 10\} \bigcup \{K_s\nabla C_5 \mid s \geq 6\} \bigcup \{K(l, s+2, k+1 \mid 2 \leq l \leq 3, s+k+l \geq 9)$. If $G = (K_s \cup K_t)\nabla K_1$ with $s, t \geq 0$ and $s+t \geq 10$, then $M_G \in \mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{H}_3 \cup \mathcal{H}_4$ due to Theorem 2. If $G = K_s\nabla C_5$, then Theorem 4 indicates $s \leq 2$, which contradict $s \geq 6$. If $G = K(l, s+2, k+1)$ with $2 \leq l \leq 3$ and $s+k+l \geq 9$, then $M_G \in \mathcal{H}_5 \cup \mathcal{H}_6$ due to Theorem 3. This completes the proof. $\square$

4. Conclusion

Let $\mathcal{M}_n(-\sqrt{3})$ be the set of all mixed graphs on n vertices with smallest eigenvalue greater than $-\sqrt{3}$. In this paper, we first determine the underlying graphs of the mixed graphs in $\mathcal{M}_n(-\sqrt{3})$, which consists of $\mathcal{G}_i$ for $0 \leq i \leq 3$ where $\mathcal{G}_0$ consists of 32 scattered graphs and $\mathcal{G}_i$ are infinite classes of graphs for $1 \leq i \leq 3$. Furthermore, we completely determined the mixed graphs in $\mathcal{M}_n(-\sqrt{3})$ when their underlying graphs belong to $\cup_{1\le i\le 3}\mathcal{G}_i$. As a result, all connected mixed graphs on $n \ge 11$ vertices with smallest eigenvalues greater than $-\sqrt{3}$ are obtained.

In fact, all mixed graphs in $\mathcal{M}_n(-\sqrt{3})$ with underlying graphs belong to $\mathcal{G}_0$ can be easily obtained with the help of computer. For example, there are about 68 mixed graphs in $\mathcal{M}_6(-\sqrt{3})$ with the underlying graph being $2K_1\nabla(K_1\cup K_3)$, and there are about 198 mixed graphs in $\mathcal{M}_8(-\sqrt{3})$ with the underlying graph being $2K_1\nabla(K_1\cup K_4)$. Therefore, we do not list them in the paper. Loosely speaking, we theoretically determine all mixed graphs with smallest eigenvalue greater than $-\sqrt{3}$.

Declaration of competing interest

There is no competing interest.

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