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Matrix-Tree Theorem of digraphs via signless Laplacians

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A B S T R A C T

Tutte's Matrix-Tree Theorem gives the number of spanning trees in a connected digraph by the determinant of a submatrix of the Laplacian, which is a generalization of the original Kirchhoff's Matrix-Tree Theorem associated to undirected graphs. In this paper, we obtain a new extension of Tutte's Matrix-Tree Theorem in terms of the signless Laplaicians. Moreover, we provide an elementary proof in terms of the factorization of signless Laplacian matrix into two incidence matrices. We finally generalize our result to the directed weighted graphs.

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1. Introduction

The number of spanning trees is an important quantity in the graph theory, networks, statistical mechanics and physical system. Historically, concerning the enumeration of spanning trees in the electrical networks, Kirchhoff [7] in 1847 discovered the celebrated Matrix-Tree Theorem (MTT), which gave the number of spanning trees based on the determinant of a submatrix of the Laplacian or, equivalently, as the product of the non-zero eigenvalues of the Laplacian.

Nowadays, Kirchhoff's Matrix-Tree Theorem has been extended to the digraphs [1,10, 3,11]. In terms of [3], the earliest proof of this extension gave the credit to [1], although the result is usually ascribed to Tutte [11] and thereby known as Tutte's Theorem. Very recently, De Leenheer [8] not only presented an elementary proof of Tutte's Theorem but also generalized it to the weighted digraphs. On the other hand, Monfared and Mallik [6] provided an analog of Kirchhoff's Matrix-Tree Theorem for signless Laplacians. See [4,5,9,12] for more new recent results about this topic. The aim of this paper is to expound Tutte's Theorem in terms of the signless Laplacians.

About the methods to show Tutte's MTT, most proofs adopt the Leibniz formula for the determinant of a matrix which can be expressed as a sum over permutations of the matrix elements to the reduced Laplacian matrix. Some proofs [10,3] rely on counting schemes that apply the inclusion-exclusion principle to collections of special subgraphs of the initial digraph. Another goal of this paper is to offer an elementary proof of Tutte's MTT for the signless Laplacians. The methods used here are to factor the signless Laplacian matrix as a product of two rectangular matrices and then use Cauchy-Binet Theorem (see Lemma 3.1). Note that both factors in this product are closely connected with the so-called incidence matrices associated with the digraphs. Moreover, the basic proof here will be helpful for us to clarify the intrinsic relations among the various matrices of digraphs, and to catch the relations between the orientations and the topological structures of graphs.

For the sake of convenience, we introduce some notations and terminology from the spectral graph theory. Let $G = (V(G), E(G))$ be an undirected simple graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $\{e_1, e_2, \dots, e_m\}$. The well-known *adjacency matrix*, denoted by $A(G) = (a_{ij})_{n\times n}$, is the $(0,1)$-symmetric matrix with $a_{ij} = 1$ if $v_iv_j \in E(G)$ and $a_{ij} = 0$ otherwise. For $v \in V(G)$, $d(v)$ denotes the degree of a vertex v and then $D(G) = \mathrm{diag}(d(v_1), d(v_2), \dots, d(v_n))$ is the degree matrix of G. The *Laplacian* and the *signless Laplacian* matrices are respectively defined as $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$. Let L^i (or Q^i) be the *reduced (signless) Laplacian* by removing the i-th row and the i-th column indexed by the vertex v_i. A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$; H is a *spanning subgraph* of G if $V(H) = V(G)$. A *spanning tree* of G is a connected subgraph of G on all n vertices with $n-1$ edges. The number of distinct spanning trees of G is denoted by $\tau(G)$ which is given by the well-known Kirchhoff's Matrix-Tree Theorem.

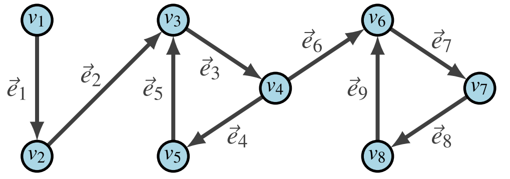

Fig. 1. An example of a digraph Γ.

Theorem 1.1 *(Kirchhoff's Matrix-Tree Theorem). Let G be a connected graph with n vertices and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \mu_n = 0$ be the eigenvalues of its Laplacian $L(G)$. Then*

$$\tau(G) = \det(L^i) = \frac{1}{n}\prod_{i=1}^{n-1} \mu_i, \tag{1}$$

for all $i = 1, 2, \ldots, n$.

We next review Tutte's Matrix-Tree Theorem which concerns the digraphs. Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a simple digraph (short for directed graphs) with vertex set $V(\Gamma) = \{v_1, \ldots, v_p\}$ and arc set $E(\Gamma) = \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_q\}$ such that an ordered pair of distinct vertices of Γ is linked by only one arc of Γ. More precisely, we assume that each arc points from some vertex v_i to another vertex $v_j \neq v_i$, and that there exists at most one arc from any vertex to any distinct vertex (see an example of a digraph in Fig. 1).

For two vertices u and v, if there is an arc $\vec{e}$ from u to v, then we write $u \to v$, where u and v are respectively the *tail* and the *head* of $\vec{e}$. We write $u \sim v$ if either $u \to v$ or $v \to u$. For a vertex v and an arc $\vec{e}$, denote by $v \to \vec{e}$ if v is the tail of $\vec{e}$ and $\vec{e} \to v$ if v is the head of $\vec{e}$. We write $v \sim \vec{e}$ if either $v \to \vec{e}$ or $\vec{e} \to v$. The *in-degree* $d^+(v)$ and *out-degree* $d^-(v)$ of v are defined as $d^+(v) = |\{\vec{e} \mid \vec{e} \to v\}|$ and $d^-(v) = |\{\vec{e} \mid v \to \vec{e}\}|$, respectively.

A *directed cycle* of a digraph Γ is a collection of distinct vertices $\{v_{i_1}, v_{i_2}, \ldots, v_{i_n}\}$ and a collection of distinct arcs $\{\vec{e}_{k_1}, \vec{e}_{k_2}, \ldots, \vec{e}_{k_n}\}$ such that each $\vec{e}_{k_i}$ points from v_{i_i} to $v_{i_{i+1}}$, and where $v_{i_{n+1}} = v_{i_1}$. The *girth* of Γ is the length of the shortest cycle in Γ; if Γ has no cycles we define the girth of Γ to be infinite. A connected digraph Γ is called an *odd-unicyclic digraph* if $|V(\Gamma)| = |E(\Gamma)|$ and its girth is odd; otherwise, Γ is an *even-unicyclic digraph*.

Example 1. Consider the digraph in Fig. 1, which we shall use as a running example throughout this paper to illustrate the various concepts and notions. This digraph has $p = 8$ vertices and $q = 9$ arcs. For example, the in-degrees of v_1, v_2, v_3 are respectively $0, 1, 2$ and the out-degrees of v_4, v_5, v_7 are severally $2, 1, 1$. Moreover, there are two directed cycles, i.e., $\{v_3, v_4, v_5\}$ and $\{\vec{e}_3, \vec{e}_4, \vec{e}_5\}$; $\{v_6, v_7, v_8\}$ and $\{\vec{e}_7, \vec{e}_8, \vec{e}_9\}$.

Recall, a *directed subgraph* of Γ is a digraph $\Gamma' = (V'(\Gamma'), E'(\Gamma'))$ with $V'(\Gamma') \subseteq V(\Gamma)$ and $E'(\Gamma') \subseteq E(\Gamma)$.

An outgoing directed spanning tree rooted at $v_r = v_1$ for the digraph from Figure 1.

An incoming directed spanning tree rooted at $v_r = v_8$ for the digraph from Figure 1.

Fig. 2. Two directed spanning trees of Γ.

Definition 1.1 *([8]).* Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a digraph with a fixed vertex $v_r \in V(\Gamma)$. Then a directed subgraph Γ' is said to be an outgoing (incoming) directed spanning tree rooted at v_r (see Fig. 2) if $V' = V$ and if the next three conditions are satisfied:

(1) Each vertex $v_i \neq v_r$ in V' has in-degree (out-degree) 1.
(2) The root vertex v_r is of in-degree (out-degree) 0.
(3) Γ' contains no directed cycles.

For a digraph Γ with p vertices, we define the following two *Laplacians* of Γ to be real matrices of order p:

$$\mathcal{L}_1 = D_{in} - \mathcal{A} \quad \text{and} \quad \mathcal{L}_2 = D_{\text{out}} - \mathcal{A}^T,$$

where $D_{in} = \text{diag}(d^+(v_1), d^+(v_2), \ldots, d^+(v_p))$ and $D_{\text{out}} = \text{diag}(d^-(v_1), d^-(v_2), \ldots, d^-(v_p))$ are diagonal matrices and $\mathcal{A}$ is the adjacency matrix of Γ, a real $p \times p$ matrix defined entrywise as follows:

$$[\mathcal{A}]_{ij} = \begin{cases} 1 \text{ if } v_i \to v_j; \\ 0 \text{ otherwise.} \end{cases}$$

For a fixed vertex v_r in Γ, we define the *reduced Laplacians* $\mathcal{L}_1^r$ and $\mathcal{L}_2^r$ by removing the r-th row and r-th column from $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. We are now ready to state the Tutte's Matrix-Tree Theorem.

Theorem 1.2 *(Tutte's Matrix-Tree Theorem). Let Γ be a connected digraph. Then*

$$\tau_{out}(\Gamma_r) = \det(\mathcal{L}_1^r) \quad \textit{and} \quad \tau_{in}(\Gamma_r) = \det(\mathcal{L}_2^r),$$

where $\tau_{out}(\Gamma_r)$ and $\tau_{in}(\Gamma_r)$ are the numbers of outgoing and incoming directed spanning trees rooted at v_r, respectively.

Similarly, we define the *signless Laplacians* of Γ as follows:

$$\mathcal{Q}_1 = D_{in} + \mathcal{A} \quad \text{and} \quad \mathcal{Q}_2 = D_{\text{out}} + \mathcal{A}^T.$$

For a given vertex v_r in Γ, we define the *reduced signless Laplacians* $\mathcal{Q}_1^r$ and $\mathcal{Q}_2^r$ by removing the r-th row and r-th column from $\mathcal{Q}_1$ and $\mathcal{Q}_2$, respectively.

Example 2. For the digraph Γ in Fig. 1, picking the root $v_r = v_1$ we have that

$$\mathcal{Q}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathcal{Q}_1^1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

A *directed TU-subgraph* of Γ is a spanning subgraph whose components are directed trees or odd-unicyclic digraphs. For a directed TU-subgraph Ω of Γ, the number of connected components that are odd-unicyclic digraphs is denoted by $\alpha(\Omega)$ (see Example 3). Let Ω_1 (or Ω_2) be a directed TU-subgraph rooted at v_r of Γ with the in-degree (or out-degree) of v_r being 0 and the others being 1's. Remark, from Lemma 2.2, that Ω_i has only one connected component that is a directed tree and $\alpha(\Omega_i)$ connected components that are odd-unicyclic digraphs ($i = 1, 2$).

Example 3. Let v_1 be the root of digraph Γ in Fig. 1. Then the directed TU-subgraphs Ω_1 of Γ are shown in Fig. 3:

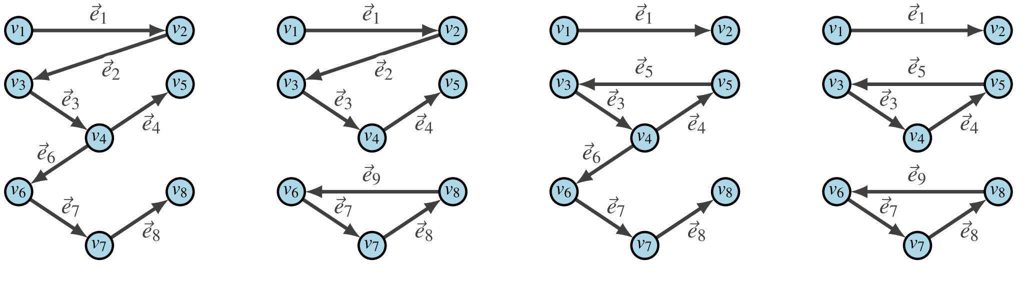

Fig. 3. The directed TU-subgraphs Ω_1 of Γ.

We are now ready to introduce the main result of this paper.

Theorem 1.3 *(Tutte's MTT for signless Laplacians). Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a connected digraph with order p and size q. Then*

Fig. 4. The directed TU-subgraphs Ω_2 of Γ.

$$\det(\boldsymbol{Q}_1^r) = \sum_{\Omega_1 \subseteq \Gamma} 2^{\alpha(\Omega_1)}, \quad \textit{and} \quad \det(\boldsymbol{Q}_2^r) = \sum_{\Omega_2 \subseteq \Gamma} 2^{\alpha(\Omega_2)},$$

where the summation runs over all directed TU-subgraphs Ω_i of Γ $(i = 1, 2)$.

Example 4. For the digraph Γ in Fig. 1, by Example 2 we have directly computed $\det(Q_1^1) = 9$. On the other hand, we calculate it in terms of the first formula in Theorem 1.3 again. For the root v_1, by the directed TU-subgraphs Ω_1 of Γ in Fig. 3 we get that

$$\det(\boldsymbol{Q}_1^1) = \sum_{\Omega_1 \subseteq \Gamma} 2^{\alpha(\Omega_1)} = 2^0 + 2^1 + 2^1 + 2^2 = 9.$$

We next adopt the second formula in Theorem 1.3 as another instance. Pick the vertex v_8 as the root in the digraph Γ. Then

$$\boldsymbol{Q}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \text{ and } \boldsymbol{Q}_2^8 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Clearly, all the directed TU-subgraphs Ω_2 of Γ are described in Fig. 4. Thereby, $\det(\boldsymbol{Q}_2^8) = \sum_{\Omega_2 \subseteq \Gamma} 2^{\alpha(\Omega_2)} = 2^0 + 2^1 = 3$.

Let $\tau_{out}(\Gamma_r)$ and $\tau_{in}(\Gamma_r)$ be defined in Theorem 1.2. Set $\tau_{out}(\Gamma) = \sum_{i=1}^{n} \tau_{out}(\Gamma_r)$, the number of all the outgoing spanning trees in Γ. Note that the following result can be similarly proved for $\tau_{in}(\Gamma_r)$.

Corollary 1.4. *Let $\lambda_1, \cdots, \lambda_n$ be the eigenvalues of the signless Laplacian $\boldsymbol{Q}_1$ of a digraph Γ. Then*

$$\det(\mathbf{Q}_1^r) \geq \tau_{out}(\Gamma_r) \quad \textit{and} \quad \sum_{1\leq i_1<\cdots<i_{n-1}\leq n} \prod_{j=1}^{n-1} \lambda_{i_j} = \sum_{r=1}^{n} \det(\mathbf{Q}_1^r) \geq \tau_{out}(\Gamma) \qquad (2)$$

where the equality holds if and only if one of the followings holds

(i) $\det(\mathbf{Q}_1^r(\Gamma)) = 0$;
(ii) $\det(\mathbf{Q}_1^r(\Gamma)) > 0$ *and each directed odd cycle contains the vertex* v_r.

The remainder of the paper is organized as follows: In Section 2 we discuss the various relations among the matrices of digraphs. In Section 3, we give an elementary proof of Tutte's MTT for signless Laplacians. Moreover, we determine a necessary and sufficient condition for $\det(\mathbf{Q}_1^r(\Gamma)) = \tau_{out}(\Gamma_r)$, and provide an example to enumerate the spanning trees in the Heawood digraph. In Section 4, we generalize our main results to the weighted digraphs.

2. Relations among the matrices of digraphs

In this section, we investigate the various relations among the matrices of digraphs, including the in-degree matrix, the out-degree matrix, the in-incidence matrix, the out-incidence matrix, the incidence matrix, the (reduced) signless Laplacian and so on.

Definition 2.1. Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a digraph. Then

(i) The in-incidence matrix $N_{in}(\Gamma)$ is the matrix whose rows are indexed by $E(\Gamma)$ and columns are indexed by $V(\Gamma)$ with $(\vec{e}, v)$-th entry being 1 if $\vec{e} \to v$ and 0 otherwise.
(ii) The out-incidence matrix $M_{out}(\Gamma)$ is the matrix whose rows are indexed by $V(\Gamma)$ and columns are index by $E(\Gamma)$ with $(v, \vec{e})$-th entry being 1 if $v \to \vec{e}$ and 0 otherwise.
(iii) The incidence matrix $N(\Gamma)$ is the matrix whose rows are indexed by $V(\Gamma)$ and columns are indexed by $E(\Gamma)$ with $(v, \vec{e})$-th entry being 1 if $v \sim \vec{e}$ and 0 otherwise.

Example 5. For the digraph from Fig. 1,

$$N_{in} = \begin{bmatrix} 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&1&0&0 \end{bmatrix} \quad \text{and} \quad M_{out} = \begin{bmatrix} 1&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&1&0&1&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&1 \end{bmatrix}.$$

The above two incidence matrices record which arcs point to, respectively, from each vertex. But on the other hand, both matrices also provide us with global information about the digraph. Indeed, given these two matrices, we can unambiguously construct the digraph. This suggests that perhaps the two signless Laplacians of a digraph can be expressed in terms of just the two incidence matrices. The following factorization result shows that this is indeed the case.

Lemma 2.1 *([8]). Let* Γ *be a digraph. Then*

$$D_{in} = N_{in}^T N_{in},\ \mathcal{A} = M_{out} N_{in},\ \textit{and } D_{out} = M_{out} M_{out}^T.$$

For the (reduced) signless Laplacians, from the above lemma we get the following equalities:

$$\mathcal{Q}_1 = (N_{in}^T + M_{out}) N_{in} \text{ and } \mathcal{Q}_2 = \left(M_{out} + N_{in}^T\right) M_{out}^T.$$

Since $N_{in}^T + M_{out} = N$, then

$$\mathcal{Q}_1 = N \cdot N_{in} \text{ and } \mathcal{Q}_2 = N \cdot M_{out}^T. \tag{3}$$

For a fixed vertex v_r in V, let N_{in}^r be the matrix obtained from N_{in} by removing the rth column in N_{in}, and let M_{out}^r be the matrix obtained from M_{out} be removing the rth row from M_{out}. Since $(N_{in}^r)^T + M_{out}^r = N^r$, then

$$\mathcal{Q}_1^r = N^r N_{in}^r \quad \text{and} \quad \mathcal{Q}_2^r = N^r \left(M_{out}^r\right)^T. \tag{4}$$

Lemma 2.2. *Let* Γ *be a connected digraph. If there exists* u *such that* $d^+(u) = 0$ *and* $d^+(v) = 1$ *for any* $v \neq u$*, then* Γ *is a tree; if* $d^+(v) = 1$ *for any* $v \in V(\Gamma)$*, then* Γ *is a unicyclic graph.*

Proof. We first show the first part. Suppose to the contrary that Γ contains a cycle C with length s. Since $\sum_{v\in C} d_C^+(v) = s$, we have $u \notin C$ and $C = v_1 \to v_2 \to \cdots \to v_s \to v_1$. Since Γ is connected, there is a shortest path from u to C, say $u \sim u_1 \sim u_2 \sim \cdots \sim u_k$ is such a path where $u_k \in C$ and $u_i \notin C$ for $1 \le i \le k-1$. Since $d^+(u) = 0$ and $d^+(u_i) = 1$ for $1 \le i \le k$, we have $u_i \to u_{i+1}$ for $0 \le i \le k-1$ where $u_0 = u$. It leads to $d^+(u_k) \ge 2$, a contradiction.

Now we prove the second part. For a vertex v, denote by $v_1 \to v_2 \to \cdots \to v_t = v$ the longest directed path pointing to v. Since $d^+(v_1) = 1$, there is a vertex w satisfying $w \to v_1$. It is clear that $w = v_i$ for some $1 \le i \le t$ since otherwise $w \to v_1 \to v_2 \to \cdots \to v$ would be a longer path pointing to v, which is impossible. Thus, Γ is not a tree and wv_1 is not a cut edge. Let Γ' be the digraph obtained from Γ by deleting the arc wv_1. It is clear that Γ' is still connected, $d_{\Gamma'}^+(v_1) = 0$ and $d_{\Gamma'}^+(v) = 1$ for any $v \neq v_1$. Therefore, the first part of the statement indicates that Γ' is a tree, and thus Γ is unicyclic. □

Lemma 2.3. *Let* Γ *be a connected digraph satisfying* $d^+(v) = 1$ *for any* v*. Then* Γ *is unicyclic and*

$$\det N(\Gamma)\det N_{in}(\Gamma) = \begin{cases} 0, & \textit{if } \Gamma \textit{ is even-unicyclic}, \\ 2, & \textit{if } \Gamma \textit{ is odd-unicyclic}. \end{cases}$$

Proof. From Lemma 2.2, Γ is unicyclic. Label the vertices as $v_1, v_2, \ldots, v_n$ and the arcs as $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ such that $\vec{e}_i$ is the arc pointing to v_i for $1 \leq i \leq n$. Therefore, $N_{in}(\Gamma) = I_n$. Furthermore, if v_i is a pendent vertex, then the determinant of $N(\Gamma)$ do not change after deleting the vertex v_i and the arc $\vec{e}_i$. It means that $\det N(\Gamma) = \det N(C)$, where C is the cycle in Γ. Note that $N(C)$ is a circulant matrix whose first row contains two 1's. Then we have $\det N(C) = 0$ if $|C|$ is even and 2 otherwise. So, the result follows. □

3. An elementary proof of Tutte's MTT for signless Laplacians

We are now fully prepared to give an elementary proof of Theorem 1.3, some ideas of which are inspired by [8]. The following is the well-known Cauchy-Binet Theorem.

Proposition 3.1 *(Cauchy-Binet Theorem [2]). Let X and Y be $m \times n$ matrices. Then*

$$\det(XY^\top) = \sum_J \det(X(J))\det(Y(J)),$$

where the sum runs over the subset J of the set of columns, and $X(J)$ (resp. $Y(J)$) denotes the square submatrix of the order m of X (resp. Y) with columns indexed by J

Proof of Theorem 1.3. A proof for the reduced signless Laplacian $\boldsymbol{Q}_1^r$ is given here; while the proof for the reduced Signless Laplacian $\boldsymbol{Q}_2^r$ is omitted, due to the similarity. From (4) it follows that

$$\det(\boldsymbol{Q}_1^r) = \det(N^r \cdot N_{in}^r).$$

For convenience we let

$$X = N^r \text{ and } Y = N_{in}^r.$$

Employing Cauchy-Binet Theorem we arrive at

$$\det(\boldsymbol{Q}_1^r) = \sum_{\substack{J\subseteq\{1,\ldots,q\} \\ |J|=p-1}} \det(X[J])\det(Y[J]),$$

where the sum runs over all the subsets J of $\{1, \ldots, q\}$ which contain $p-1$ elements. Clearly, there exists $\begin{pmatrix} q \\ p-1 \end{pmatrix}$ such subsets. Moreover, let $X[J]$ denote the $(p-1)\times(p-1)$ submatrix obtained from the $(p-1)\times q$ matrix X by choosing exactly those columns of X in the set J. Analogously $Y[J]$ is the $(p-1)\times(p-1)$ submatrix which is obtained from the $q \times (p-1)$ matrix Y by choosing accurately those rows of Y in J.

To complete the conclusion, we will prove the following three claims.

Claim 1. *If the $p-1$ elements in the set J are consistent with the indices of the arcs of an outgoing directed spanning tree rooted at v_r, then* $\det(X[J])\det(Y[J]) = 1$.

Proof of Claim 1. Let $J = \{l_1, l_2, \ldots, l_{p-1}\}$ $(l_1 < l_2 < \cdots < l_{p-1})$ be in a bijective correspondence to a set of $p-1$ indices of the arcs $\vec{e}_{l_1}, \vec{e}_{l_2}, \ldots, \vec{e}_{l_{p-1}}$ of an outgoing directed spanning tree rooted at v_r. Analogously, set $\tilde{J} = \{t_1, t_2, \ldots, t_{p-1}\}$ $(t_1 < t_2 < \cdots < t_{p-1})$ to be in a bijective which is relevant with the set of indices of the vertices in the set $V \backslash \{v_r\}$. We denote by $T = (V, E')$ the directed subgraph of Γ which is related to this tree, that is, $E' = \{\vec{e}_{l_1}, \vec{e}_{l_2}, \ldots, \vec{e}_{l_{p-1}}\}$.

It is clear that for an outgoing directed spanning tree rooted at v_r, the in-degree of every vertex other than v_r is 1, which indicates that each column of $Y[J] = N^r_{in}[J]$ has only one nonzero entry equal to 1. Moreover, any one of the $p-1$ arcs points to one of the $p-1$ non-root vertices, which shows that each row of $Y[J] = N^r_{in}[J]$ has merely one nonzero entry which is 1. Consequently, the $(p-1) \times (p-1)$ matrix $Y[J] = N^r_{in}[J]$ contains exactly one nonzero entry in each row and in each column, which is equal to 1.

Thereby, the matrix $Y[J] = N^r_{in}[J]$ is a permutation matrix implying $Y[J](Y[J])^T = (Y[J])^T Y[J] = I$, which leads to

$$\begin{aligned}\det(X[J])\det(Y[J]) &= \det(X[J]X[J]) \\ &= \det(I + M^r_{\text{out}}[J]N^r_{in}[J]) \\ &= \det(I + D)\end{aligned} \tag{5}$$

where

$$D = M^r_{out}[J]N^r_{in}[J].$$

Since D is nilpotent [8, the second claim in pp. 722], and then there exists an invertible $(p-1) \times (p-1)$ matrix R satisfying $R^{-1}DR = \mathcal{J}$, where the Jordan canonical form $\mathcal{J}$ of D is strictly upper-triangular. Along with (5) we get

$$\det(X[J])\det(Y[J]) = \det(I + D) = \det(R(I + \mathcal{J})R^{-1}) = \det(I + \mathcal{J}) = 1.$$

Claim 2. *If the $p-1$ elements in the set J are corresponding to the indices of the arcs of Ω_1 rooted at v_r, then* $\det(X[J])\det(Y[J]) = 2^{\alpha(\Omega_1)}$.

Proof of Claim 2. Suppose that $J = \{l_1, \ldots, l_{p_0}\} \cup \{l_{p_0+1}, \ldots, l_{p_1}\} \cup \cdots \cup \{l_{p_{\alpha(\Omega_1)}+1}, \ldots, l_{p-1}\}$ $(l_1 < l_2 < \cdots < l_{p-1})$ is in a bijective which is correlative with a set of $p-1$ indices of the arcs $E_0 \cup E_1 \cup \cdots \cup E_{\alpha(\Omega_1)}$, where $E_0 = \{\vec{e}_{l_1}, \ldots, \vec{e}_{l_{p_0}}\}$, $E_1 = \{\vec{e}_{l_{p_0+1}}, \ldots, \vec{e}_{l_{p_1}}\}, \cdots, E_{\alpha(\Omega_1)} = \{\vec{e}_{l_{p_{\alpha(\Omega_1)}+1}}, \ldots, \vec{e}_{l_{p-1}}\}$. Recall that all except one connected components of Ω_1 are directed odd-unicyclic graphs and the other one is a directed tree. Then we can set $\Gamma[E_0]$ to be a directed tree, $\Gamma[E_i]$ $(1 \le i \le \alpha(\Omega_1))$ is a directed odd-unicyclic graph. Hence,

$$\det(X[J](\Gamma)) = \det \begin{bmatrix} N^r(\Gamma[E_0]) & & & \\ & N(\Gamma[E_1]) & & \\ & & \ddots & \\ & & & N(\Gamma[E_\alpha]) \end{bmatrix}$$

and

$$\det(Y[J](\Gamma)) = \det \begin{bmatrix} N^r_{in}(\Gamma[E_0]) & & & \\ & N_{in}(\Gamma[E_1]) & & \\ & & \ddots & \\ & & & N_{in}(\Gamma[E_\alpha]) \end{bmatrix}.$$

So, by Lemma 2.3 and Claim 1 we get

$$\begin{aligned}\det(X[J])\det(Y[J]) &= \det((N^r N^r_{in})(\Gamma[E_0]))\prod_{i=1}^{\alpha}\det(N(\Gamma(E_i))\det(N_{in}(\Gamma(E_i)) \\ &= 2^{\alpha(\Omega_1)}.\end{aligned}$$

Claim 3. *If the* $p-1$ *elements in the set* J *are associated to the indices of the arcs of a directed subgraph of* Γ *which is not the TU-subgraph* Ω_1 *or an outgoing directed spanning tree rooted at* v_r*, then* $\det(X[J])\det(Y[J]) = 0$.

Proof of Claim 3. In this case, set $J = \{l_1, \ldots, l_{p-1}\}$ $(l_1 < \cdots < l_{p-1})$ to correspond to the index set of a subgraph $\Gamma' = (V, E')$ of Γ with $E' = \{\vec{e}_{l_1}, \ldots, \vec{e}_{l_{p-1}}\}$, which is neither Ω_1 nor an outgoing directed spanning tree rooted at v_r. We next prove that

$$\det(X[J])\det(Y[J]) = 0.$$

By Definition 1.1 we consider three possible cases.

Case 1. There exists a vertex v_i in Γ' with $v_i \neq v_r$, with in-degree not bing 1. Thus, $d^+(v_i) = 0$ or $d^+(v_i) \ge 2$. For the former case, the column of $N^r_{in}[J]$ that represents all in-coming arcs to v_i in Γ' is a zero column vector, and so $\det(X[J]) = \det(N^r_{in}[J]) = 0$. For the latter case, we have at least two identical rows in the matrix $N^r_{in}[J]$, and similarly $\det(Y[J]) = \det(N^r_{in}[J]) = 0$.

Case 2. The in-degree of the root v_r is not 0. Hence $N^r_{in}[J]$ contains at least one zero row, and thereby $\det(Y[J]) = \det(N^r_{in}[J]) = 0$.

Case 3. Γ' contains a directed cycle. By Cases 1 and 2 we can set $d^+(v_r) = 0$ and $d^+(v) = 1$ for any $v \neq v_r$. Therefore, each component of Γ' not containing v_r is a unicyclic graph, and the component containing v_r is a tree by Lemma 2.2. Since $\Gamma' \ncong \Omega_1$, then by $V(\Gamma') = |E'| + 1$ we get that Γ' contains an even-unicyclic graph. From Lemma 2.3 it follows that $\det(X[J]) = 0$. $\square$

Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of matrix $M_{n\times n}$. The *k-th symmetric function* s_k of $\lambda_1, \lambda_2, \cdots, \lambda_n$ is defined to be the sum of the product of the eigenvalues taken k at a time. In other words,

$$s_k = \sum_{1\le i_1<\cdots<i_k\le n} \lambda_{i_1}\cdots\lambda_{i_k}.$$

In [13, pp. 494], the following statement has been introduced.

Lemma 3.2. *$s_k = \sum M_{k\times k}$, where $\sum M_{k\times k}$ is the sum of all $k\times k$ principal minors of $M_{n\times n}$.*

We are now prepared to prove Corollary 1.4.

Proof of Corollary 1.4. Setting $k = n-1$ in Lemma 3.2, we get the first equality of the latter statement in (2). So, the remaining work is to show the equality of the former inequality in (2).

Firstly, we show the sufficiency. There is nothing to prove if (i) holds, and thereby we assume $\det(\boldsymbol{Q}_1^r(\Gamma)) > 0$. For any $\Omega_1 \subseteq \Gamma$, since each odd directed cycle contains v_r and the component of Ω_1 containing v_r is a tree, Ω_1 is a spanning tree rooted at v_r, and thus $\det(\boldsymbol{Q}_1^r(\Gamma)) = \tau_{out}(\Gamma_r)$ by Theorem 1.3.

Conversely, if $\det(\boldsymbol{Q}_1^r(\Gamma)) = \tau_{out}(\Gamma_r)$, then either Γ contains no TU-subgraph Ω_1 or each Ω_1 is a spanning tree rooted on v_r. For the former case, we have $\det(\boldsymbol{Q}_1^r(\Gamma)) = 0$ and (i) follows. Otherwise, assume that $\Omega_1 \in \Gamma$. If there is a directed odd cycle C not containing v_r, then there is a path $v_1 \to v_2 \to \cdots \to v_s$ in Ω_1 such that $C = v_1 \to v_2 \to \cdots \to v_s \to v_1$. Suppose that $\vec{e}$ is the arc in Ω_1 such that $\vec{e} \to v_1$. Let $\Omega_1' = \Omega_1 - \vec{e} + v_s v_1$ be the graph obtained from Ω_1 by deleting $\vec{e}$ and adding $v_s v_1$. It is clear $\Omega_1' \in \Gamma$ is not a spanning tree, a contradiction. $\square$

Algorithm 1 Induce $S(X)$ from X.

Require: $X \subseteq \Gamma$
Ensure: Obtain $S(X)$
  $S(X) \leftarrow X$
  if $\exists v \notin X$ and $u \in X$ such that $u \to v$ **then**
    $S(X) \leftarrow X + v + uv$
  end if

Though Corollary 1.4 gives a sufficient and necessary condition for $\det Q_1^r(\Gamma) = \tau_{out}(\Gamma_r)$, the condition (i) is not clear enough. Next we present an arithmetic to de-

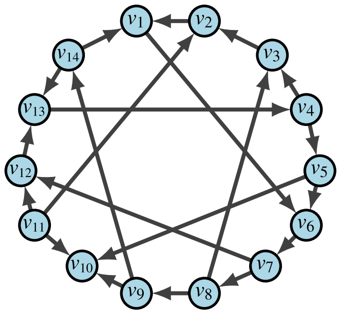

Fig. 5. The Heawood digraph.

termine whether $\det Q_1^r(\Gamma) = 0$. For a subgraph X of Γ, we call $S(X)$ obtained by Algorithm 1 a out-stretched subgraph.

Then we get the reminder-graph $R(\Gamma, v_r)$ from Γ and a vertex v_r by Algorithm 2.

Algorithm 2 Induce $R(\Gamma, v_r)$ from Γ and v_r.

Require: Digraph Γ and vertex v_r
Ensure: Obtain $R(\Gamma, v_r)$
  $R(\Gamma, r) \leftarrow \Gamma - S(\{v_r\})$
  if $R(\Gamma, v_r)$ contains an odd directed cycle C **then**
    $R(\Gamma, v_r) \leftarrow R(\Gamma, v_r) - S(C)$
  end if

Clearly, $\det(\mathcal{Q}_1^r(\Gamma)) \neq 0$ if and only if Γ has an outgoing spanning tree rooted at v_r, or Γ contains a directed TU-subgraph Ω_1 rooted at v_r. For the former case, by Algorithm 1 we get that $R(\Gamma, v_r)$ is null; for the latter case, by Algorithms 1 and 2 we also get $R(\Gamma, v_r)$ to be null. Thus, the following corollary follows.

Corollary 3.3. $\det(\mathcal{Q}_1^r(\Gamma)) = 0$ *if and only if* $R(\Gamma, r)$ *is not null.*

Let the characteristic polynomial of matrix $M_{n\times n}$ be $f_M(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_1\lambda + a_0$. Then by the well-known Vieta's Theorem we get the k-th symmetric function s_k to be

$$s_k = (-1)^k \frac{a_{n-k}}{a_n}. \tag{6}$$

Example 6. In the end of this section, we use Corollary 1.4 to enumerate all spanning trees in the Heawood digraphs in Fig. 5. Clearly, the signless Laplacian Q_1 of Heawood digraph is shown as follows:

$$
\boldsymbol{Q}_1 = \begin{bmatrix}
2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{bmatrix}.
$$

A direct calculation shows that the characteristic polynomial of $\boldsymbol{Q}_1$ is

$$
\begin{aligned}
f_{\boldsymbol{Q}_1}(\lambda) = \lambda^{14} - 21\lambda^{13} + 202\lambda^{12} - 1180\lambda^{11} + 4678\lambda^{10} - 13318\lambda^{9} + 28084\lambda^{8} - 44547\lambda^{7} \\
+ 53327\lambda^{6} - 47743\lambda^{5} + 31173\lambda^{4} - 14098\lambda^{3} + 3967\lambda^{2} - 525\lambda,
\end{aligned}
$$

with the coefficients $a_{14} = 1$ and $a_1 = -525$ as well as the eigenvalues

$$3, 2.9253, 1.3269, 1, 1, 0, 2.1665 \pm 0.8049i, 2.0707 \pm 0.7687i, 0.9293 \pm 0.7587i, 0.7073 \pm 0.8398i.$$

In terms of Corollary 1.4(ii) and (6), we get

$$
\tau_{out}(\Gamma) = \Sigma_{i=1}^{r} \tau_{out}(\Gamma_r) = \sum_{1 \leq i_1 < \cdots < i_{13} \leq 14} \prod_{j=1}^{13} \lambda_{i_j} = s_{13} = (-1)^{13} \frac{a_1}{a_{14}} = 525.
$$

4. Extension to weighted digraphs

Let $\Gamma_w = (V, E, W)$ be a weighted digraph, where $V = \{v_1, \cdots, v_p\}$ is the vertex set, $E = \{\vec{e}_1, \cdots, \vec{e}_q\}$ the arc set, and $W = \{w_1, \cdots, w_q\}$ is the set positive weights associated to each of arcs, i.e., $w_i = w(\vec{e}_i)$. For a subgraph Γ'_w of Γ_w, we denote by $w(\Gamma'_w) = \prod_{\vec{e} \in E(\Gamma'_w)} w(\vec{e})$ the weight of Γ'_w.

For a vertex $v \in V$, the weighted in-degree $d_w^+(v)$ and the weighted out-degree $d_w^-(v)$ are defined as $d_w^+(v) = \sum_{e \to v} w(\vec{e})$ and $d_w^-(v) = \sum_{v \to \vec{e}} w(\vec{e})$, respectively. The in-degree (or out-degree) matrix $\mathscr{D}_{in}$ (or $\mathscr{D}_{out}$) is the diagonal matrix indexed by $V(\Gamma_w)$ whose v-th diagonal entry is $d_w^+(v)$ (or $d_w^-(v)$). The adjacency matrix $\mathscr{A}$ is the square matrix indexed by $V(\Gamma_w)$ whose (u, v)-th entry is $w(uv)$ if $u \to v$ and 0 otherwise.

The in-incidence matrix $\mathscr{N}_{in}$ is the matrix whose rows are indexed by $E(\Gamma_w)$ and columns are indexed by $V(\Gamma_w)$ with $(\vec{e}, v)$-th entry being $\sqrt{w(\vec{e})}$ if $\vec{e} \to v$ and 0 otherwise. The out-incidence matrix $\mathscr{M}_{out}$ is the matrix whose rows are indexed by $V(\Gamma_w)$ and columns are indexed by $E(\Gamma_w)$ with $(v, \vec{e})$-th entry being $\sqrt{w(\vec{e})}$ if $v \to \vec{e}$ and 0 otherwise.

The incidence matrix $\mathscr{N}$ is the matrix whose rows are indexed by $V(\Gamma_w)$ and columns are indexed by $E(\Gamma_w)$ with $(v, \vec{e})$-th entry being $\sqrt{w(\vec{e})}$ if $v \sim \vec{e}$ and 0 otherwise. Then signless Laplacians of Γ_w are defined to be

$$\mathscr{Q}_1 = \mathscr{D}_{in} + \mathscr{A} \quad \text{and} \quad \mathscr{Q}_2 = \mathscr{D}_{out} + \mathscr{A}^T.$$

Lemma 4.1. *$\mathscr{Q}_1 = \mathscr{N}\mathscr{N}_{in}$ and $\mathscr{Q}_2 = \mathscr{N}\mathscr{M}_{out}$.*

Proof. By immediate calculations, the (u, v)-th entry of $\mathscr{N}\mathscr{N}_{in}$ is

$$(\mathscr{N}\mathscr{N}_{in})_{uv} = \sum_{\vec{e}} (\mathscr{N})_{u\vec{e}} (\mathscr{N}_{in})_{\vec{e}v} = \sum_{u\sim\vec{e}} (\mathscr{N})_{u\vec{e}} (\mathscr{N}_{in})_{\vec{e}v} = \sum_{\substack{u\sim\vec{e} \\ \vec{e}\to v}} w(\vec{e}).$$

Then, $(\mathscr{N}\mathscr{N}_{in})_{uv} = d_w^+(u)$ if $u = v$, $(\mathscr{N}\mathscr{N}_{in})_{uv} = w(uw)$ if $u \to v$ and $(\mathscr{N}\mathscr{N}_{in})_{uv} = 0$ otherwise. Therefore we are done. $\square$

Naturally, we can extend the definitions of TU-subgraphs Ω, Ω_1 and Ω_2 to the weighted TU-subgraphs Ω^w, Ω_1^w and Ω_2^w. Denote by $\alpha(\Omega_w)$ the number of unicyclic graphs in Ω^w.

Let $\mathscr{Q}_1^r$ (or $\mathscr{Q}_2^r$) be the matrix obtained by deleting the r-th row and the r-th column of $\mathscr{Q}_1$ (or $\mathscr{Q}_2$). By $\mathscr{N}^r$ we denote the matrix obtained from $\mathscr{N}$ by deleting the r-th row. Set $\mathscr{N}_{in}^r$ to be the matrix obtained from $\mathscr{N}_{in}$ by deleting the r-th column, and $\mathscr{M}_{out}^r$ to be the matrix obtained from $\mathscr{M}_{out}$ by deleting the r-th row.

Theorem 4.2. *Let Γ_w be a weighted digraph. Then*

$$\det(\mathscr{Q}_1^r) = \sum_{\Omega_1^w \subseteq \Gamma_w} 2^{\alpha(\Omega_1^w)} w(\Omega_1^w) \quad \text{and} \quad \det(\mathscr{Q}_2^r) = \sum_{\Omega_2^w \subseteq \Gamma_w} 2^{\alpha(\Omega_2^w)} w(\Omega_2^w),$$

where the summation runs over all the TU-graphs Ω_i^w of Γ_w ($i = 1, 2$).

Proof. We here only provide a proof for $\mathscr{Q}_1^r$, since the proof for $\mathscr{Q}_2^r$ is analogous. By Lemmas 4.1 and 3.1, we have

$$\det(\mathscr{Q}_1^r) = \sum_S \det(\mathscr{N}^r[J]) \det(\mathscr{N}_{in}^r[J]),$$

where $S \subseteq E(\Gamma_w)$ with $|J| = |V(\Gamma_w)| - 1$, $\mathscr{N}^r[J]$ and $\mathscr{N}_{in}^r[J]$ are the square submatrices of $\mathscr{N}$ and $\mathscr{N}_{in}$ by deleting the rows not in S. Denote by Γ'_w the subgraph with $V(\Gamma'_w) = V(\Gamma_w)$ and $E(\Gamma'_w) = J$.

By the definition of $\mathscr{N}_{in}$, if there exists $v \in V\backslash\{v_r\}$ such that $d_w^+(v) = 0$, then the v-th column of $\mathscr{N}_{in}^r$ is 0. If there exists $v \in V\backslash\{v_r\}$ such that $d_w^+(v) \geq 2$, say $\vec{e}_1, \vec{e}_2 \to v$, then the $\vec{e}_1$-th row and the $\vec{e}_2$-th row are linear dependent. If $d_w^+(r) \geq 1$, say $\vec{e} \to v_r$, then the $\vec{e}$-th row is 0. Thus, $\det(\mathscr{N}_{in}^r) \neq 0$ implies $d_w^+(r) = 0$ and $d_w^+(v) = 1$ for any $v \neq v_r$. Therefore, each component of Γ'_w not containing v_r is a unicyclic graph, and the component containing v_r is a tree due to Lemma 2.2. Note that, for any $v \in V\backslash\{v_r\}$ with $\vec{e} \to v$, we have $\det(\mathscr{N}^r_{in}) = \det(M)\sqrt{w(\vec{e})}$, where M is obtained from $\mathscr{N}^r_{in}$ by deleting the v-th row and $\vec{e}$-th column. Therefore, by repeating the deletions, we have $\det(\mathscr{N}^r_{in}) = \sqrt{w(\Omega^w_1)}$.

By the definition of $\mathscr{N}$, if there is a pendent vertex v such that $v \sim \vec{e}$, then $\det \mathscr{N} = \sqrt{w(\vec{e})} \det N'$ where N' is obtained from $\mathscr{N}$ by deleting its v-th row and $\vec{e}$-th column. Let $\Gamma'_w = \Gamma'_{w,1} \cup \Gamma'_{w,2} \cup \cdots \cup \Gamma'_{w,s}$. Denote by $\mathscr{N}^r[\Gamma'_{w,i}]$ the square submatrix corresponding to $\Gamma'_{w,i}$. Clearly, $\det \mathscr{N}^r = \prod_i \det \mathscr{N}^r[\Gamma'_{w,i}]$. Denote by C_i the directed cycle (if exists) in $\Gamma'_{w,i}$. Then we have $\det \mathscr{N}^r[C_i] = 2\sqrt{w(C_i)}$ if C_i is an odd and 0 otherwise. Therefore, we obtain $\det \mathscr{N}^r[\Gamma'_{w,i}] = w\left(E\left(\Gamma'_{w,i}\right) \setminus\ E\left(C_i\right)\right) \det \mathscr{N}^r[C_i] = 2w(\Gamma'_{w,i})$ if C_i is odd and 0 otherwise. Consequently, $\det(\mathscr{N}^r) = 2^{\alpha(\Gamma'_{w,i})}\sqrt{w(\Gamma'_{w,i})}$ if $\Gamma'_{w,i} \in \Omega^w_1$ and 0 otherwise.

This completes the proof. $\square$

Declaration of competing interest

There is no conflict of interest in this article.

Data availability

No data was used for the research described in the article.

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