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On $\mathbb{T}_4$-gain graphs with few positive eigenvalues

Xiaocong He, Lihua Feng and Lu Lu

School of Mathematics and Statistics, HNP-LAMA, Central South University, Changsha, Hunan, People's Republic of China

ABSTRACT

Let $\mathbb{T}_4 = \{1, -1, \mathbf{i}, -\mathbf{i}\}$ be the group of fourth roots of unit. A $\mathbb{T}_4$-gain graph is a graph where each orientation of an edge is given a complex unit in $\mathbb{T}_4$, which is the inverse of the complex unit assigned to the opposite orientation. In this paper, we characterize the structure of the $\mathbb{T}_4$-gain graphs with exactly one positive eigenvalue and determine the $\mathbb{T}_4$-gain graphs with cut vertices having exactly two positive eigenvalues. Our results extend some parallel ones about mixed graphs and signed graphs.

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1. Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$, and let $\mathbb{T}_4 = \{1, -1, \mathbf{i}, -\mathbf{i}\}$ be the group of fourth roots of unit. For any number $z \in \mathbb{C}$, denote by z^* the conjugate of z, and $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ the *real part* and the *imaginary part* of z, respectively. The $\mathbb{T}_4$-gain graph $\Phi = (G, \mathbb{T}_4, \varphi)$ is a graph with additional structure φ: $\vec{E}(G) \to \mathbb{T}_4$ such that $\varphi(uv) = \varphi(vu)^{-1}$ for any $\{u, v\} \in E(G)$, where $\vec{E}(G) = \{uv, vu \mid \{u, v\} \in E(G)\}$ is the set of oriented edges. The graph G is the *underlying graph* of Φ and the function φ is the *gain function* of Φ. Each simple graph can be seen as a $\mathbb{T}_4$-gain graph where the gain of each edge (if there exists any) is 1. The *adjacency matrix* of Φ is an $n \times n$ matrix $A(\Phi) = [a_{ij}]$ defined by $a_{ij} = \varphi(v_i v_j)$ if $\{v_i, v_j\} \in E(G)$ and 0 otherwise. Clearly, $A(\Phi)$ is a Hermitian matrix. Thus, the eigenvalues of $A(\Phi)$ are real. The eigenvalues of $A(\Phi)$ are called the *eigenvalues* of Φ. The multiset of all eigenvalues together with their multiplicities is the *spectrum* of Φ, denoted by $\mathrm{Sp}(\Phi)$, that is $\mathrm{Sp}(\Phi) = \{\lambda_1^{[m_1]}, \lambda_2^{[m_2]}, \ldots, \lambda_k^{[m_k]}\}$, where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are all the distinct eigenvalues and m_j is the multiplicity of the eigenvalue λ_j $(1 \le j \le k)$. The number of positive, negative and zero eigenvalues of Φ are defined as *positive inertia index, negative inertia index* and *nullity* of Φ, denoted by $p(\Phi)$, $n(\Phi)$ and $\eta(\Phi)$, respectively. The *rank* $\mathrm{rank}(\Phi)$ of Φ is the number of non-zero eigenvalues of Φ, i.e. $p(\Phi) + n(\Phi) = \mathrm{rank}(\Phi)$.

For $u \in V(G)$, we denote its neighbourhood and its degree by $N_G(u)$ and $d_G(u)$, respectively. A *pendant vertex* is the vertex of degree 1 and a *cut vertex* of G is a vertex whose removal increases the number of connected components of G. Let $\Phi = (G, \mathbb{T}_4, \varphi)$ be

CONTACT Lu Lu lulugdmath@163.com School of Mathematics and Statistics, HNP-LAMA, Central South University, Changsha, Hunan 410083, People's Republic of China

a $\mathbb{T}_4$-gain graph. If v is a cut vertex of G, we also say v is a cut vertex of Φ. For an induced subgraph H of G, the corresponding induced $\mathbb{T}_4$-gain graph, denoted by $\Phi[H]$, is defined as $\Phi[H] = (H, \mathbb{T}_4, \varphi')$ satisfying $\varphi'(uv) = \varphi(uv)$ for every $uv \in \vec{E}(H)$. If $V(H) = \{v_1, v_2, \ldots, v_m\}$, then we also write $\Phi[v_1, v_2, \ldots, v_m]$ for $\Phi[H]$. It is clear that the adjacency matrix $A(\Phi[H])$ is just the principal submatrix of $A(\Phi)$ induced by the vertex set of $V(H)$. Let $v \in V(G)$, we write $\Phi - v$ for the induced subgraph obtained from Φ by deleting the vertex v and all edges incident with v. For an induced subgraph $\Phi[H]$ of Φ and a vertex $v \in V(G)\backslash V(H)$, denote by $\Phi[H] + v$, the induced subgraph of Φ with vertex set $V(H) \cup \{v\}$. Throughout this paper, we always write K_n, $K_{n_1,n_2,\ldots,n_k}$ and P_n for the *complete graph*, the *complete multipartite graph* and the *path* respectively. Denote by I_n the identity matrix of order n. Let $\mathbf{J}_{n\times m}$ and $\mathbf{0}_{n\times m}$ be respectively the all-one and the all-zero $n \times m$ matrices. Let $\mathbf{1}_n = \mathbf{J}_{n\times 1}$ and $\mathbf{0}_n = \mathbf{0}_{n\times 1}$. If the size of these matrices are clear from the context, we often delete the subscripts.

The *converse* $\Phi^\top = (G, \mathbb{T}_4, \varphi^\top)$ of a $\mathbb{T}_4$-gain graph $\Phi = (G, \mathbb{T}_4, \varphi)$ is the $\mathbb{T}_4$-gain graph with $\varphi^\top(uv) = \varphi(uv)^{-1}$ for any $\{u, v\} \in E(G)$. Clearly, $A(\Phi^\top) = A(\Phi)^\top$, and thus Φ and $\Phi^\top$ are cospectral. Two $\mathbb{T}_4$-gain graphs $\Phi = (G, \mathbb{T}_4, \varphi)$ and $\Phi' = (G, \mathbb{T}_4, \varphi')$ are *switching equivalent* if there is a map $\theta : V(G) \to \mathbb{T}_4$ such that either $\varphi'(uv) = \theta(u)^{-1}\varphi(uv)\theta(v)$ for all $\{u, v\} \in E(G)$ or $\varphi'(uv) = \theta(u)^{-1}\varphi^\top(uv)\theta(v)$ for all $\{u, v\} \in E(G)$. In this case, the map θ is called a *switching equivalent transformation* from Φ to Φ'. Let $D = diag(\theta(v_1), \theta(v_2), \ldots, \theta(v_n))$. Then either $A(\Phi') = D^{-1}A(\Phi)D$ or $A(\Phi') = D^{-1}A(\Phi^\top)D$ and thus Φ and Φ' share the same spectrum. It is clear that the switching equivalence is an equivalence relation, and let $[\Phi]$ denote the equivalence class containing Φ.

Note that a *mixed graph* $\widetilde{G}$ (resp. *signed graph* $\Gamma(G)$) is a $\mathbb{T}_4$-gain graph in which only 1 and $\pm\mathbf{i}$ (resp. ± 1) gains are used. Hence, $\mathbb{T}_4$-gain graphs can be seen as the generalizations of mixed graphs and signed graphs. For more advances on mixed graphs and signed graphs, we refer the reader to [1–8], for examples.

The study of graphs with few eigenvalues has attracted much attention. With respect to simple graphs, Smith [9] characterized all simple graphs with exactly one positive eigenvalue; Oboudi [10] completely characterized the simple graphs with exactly two non-negative eigenvalues. With respect to signed graphs, Yu et al. [7] determined the signed graphs with exactly one positive eigenvalue and the signed graphs containing pendant vertices with exactly two positive eigenvalues; X.L. Wang et al. [5] extended the above work to the signed graphs containing cut vertices with exactly two positive eigenvalues. With respect to mixed graphs, Wissing and van Dam [3] characterized all mixed graphs with exactly one negative eigenvalue; Wei et al. [4] investigated relations between the number of positive (negative) eigenvalues of a mixed graph and those of its underlying graph; Yuan et al. [8] provided a characterization of mixed graphs with exactly one positive eigenvalue, and studied some classes of mixed graphs determined by their H-spectra. With respect to gain graphs, Lu et al. [11] characterized the structure of complex unit gain graphs with exactly one positive eigenvalue and investigated the complex unit gain graphs with exactly two eigenvalues different from 0 and -1; Zaman et al. [12] investigated relations between the number of positive (negative) eigenvalues of a complex unit gain graph and those of its underlying graph. Motivated by such works, in this paper, we characterize the structure of the $\mathbb{T}_4$-gain graphs with exactly one positive eigenvalue and determine the $\mathbb{T}_4$-gain graphs with cut vertices having exactly two positive eigenvalues.

2. Preliminaries

Firstly, we define a graph which will play an important role in the paper. Let $V_1, V_2, \ldots, V_k$ be the colour sets of $K_{n_1,n_2,\ldots,n_k}$. If there exist $\eta_{ij} \in \mathbb{T}_4$ for $1 \le i < j \le k$ such that the $\mathbb{T}_4$-gain graph $\Phi = (K_{n_1,n_2,\ldots,n_k}, \mathbb{T}_4, \varphi)$ satisfies:

(a) $\varphi(v_s v_t) = \eta_{st}$ for any $v_s \in V_s, v_t \in V_t$ and $1 \le s < t \le k$;
(b) $\eta_{1t} = 1$ for any $2 \le t \le k$,

then Φ is called a standard k-partite graph, denoted by $(K_{n_1,n_2,\ldots,n_k}; [\eta_{st} \mid 2 \le s < t \le k])$. Clearly, the simple graph $K_{n_1,n_2,\ldots,n_k}$ is a standard k-partite graph, namely, $K_{n_1,n_2,\ldots,n_k} = (K_{n_1,n_2,\ldots,n_k}; [1, 1, \ldots, 1])$. In particular, we write $\vec{K}_{n_1,n_2,n_3}$ for $(K_{n_1,n_2,n_3}; [\mathbf{i}])$.

Let u, v be distinct vertices of $\Phi = (G, \mathbb{T}_4, \varphi)$. If $N_G(u) = N_G(v)$ and $\varphi(uw) = \xi\varphi(vw)$ for all $w \in N_G(u)$ with $\xi \in \{1, -1, \mathbf{i}, -\mathbf{i}\}$, we say that u and v are *twin points* and denoted this relation by $u\mathcal{R}v$. It is easy to see that $\mathcal{R}$ is an equivalence relation on $V(G)$. A $\mathbb{T}_4$-gain graph $\Phi = (G, \mathbb{T}_4, \varphi)$ is said to be *reduced* if, for each vertex v in Φ, v has no twin points. One can easily see that deleting a twin point of a given vertex from a graph does not change its inertia indices. In fact, we will see that such operation is invertible. To see this, we first introduce a graph transformation. Given a graph G with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and a vector $\mathbf{m} = (m_1, m_2, \ldots, m_n)$ whose components are positive integers, we denote by $G \circ \mathbf{m}$ the graph obtained from G by replacing each vertex v_i of G with an independent set of m_i vertices $v_i^1, v_i^2, \ldots, v_i^{m_i}$ and joining v_i^s with v_j^t if and only if $\{v_i, v_j\} \in E(G)$. For a $\mathbb{T}_4$-gain graph $\Phi = (G, \mathbb{T}_4, \varphi)$, let $\Phi \circ \mathbf{m} = (G \circ \mathbf{m}, \mathbb{T}_4, \varphi')$ be a $\mathbb{T}_4$-gain graph with $\varphi'(v_i^s v_j^t) = \varphi(v_i v_j)$ for any $\{v_i, v_j\} \in E(G)$, $1 \le s \le m_i$ and $1 \le t \le m_j$. Now we get the following result.

Lemma 2.1: *Let $\Phi = (G, \mathbb{T}_4, \varphi)$ be a $\mathbb{T}_4$-gain graph. Then there exists a reduced $\mathbb{T}_4$-gain graph $\Phi' = (G', \mathbb{T}_4, \varphi')$ having the same inertia indices as that of Φ, and a vector $\mathbf{m}$ such that $\Phi \in [\Phi' \circ \mathbf{m}]$.*

Proof: If Φ is reduced, then let $\Phi' = \Phi$ and the result holds with Φ' and $\mathbf{m} = (1, 1, \ldots, 1)$. If Φ is not reduced, then there is an equivalence partition $V(G) = V_1 \cup V_2 \cup \cdots \cup V_p$ with respect to $\mathcal{R}$. Assume $|V_j| = m_j$ and $V_j = \{v_j^1, v_j^2, \ldots, v_j^{m_j}\}$ for $1 \le j \le p$. By the definition of twin points, for each $1 \le j \le p$ and $1 \le k \le m_j$, there is a $\xi_{jk} \in \{1, -1, \mathbf{i}, -\mathbf{i}\}$ such that $\varphi(v_j^1 w) = \xi_{jk}\varphi(v_j^k w)$ for all $w \in N_G(v_j^1)$. Especially, $\xi_{j1} = 1$. Let $\hat{\Phi} = (G, \mathbb{T}_4, \varphi_1)$ be a $\mathbb{T}_4$-gain graph with $\varphi_1(v_j^k v_s^t) = \varphi_1(v_j^1 v_s^t) = \varphi(v_j^1 v_s^t)\xi_{st}^{-1}$ for any $v_s^t \in N_G(v_j^1)$, $1 \le j \le p$ and $1 \le k \le m_j$. Note that $\xi_{jk}\varphi(v_j^k v_s^t)\xi_{st}^{-1} = \xi_{j1}\varphi(v_j^1 v_s^t)\xi_{st}^{-1} = \varphi_1(v_j^1 v_s^t)$ for any $v_s^t \in N_G(v_j^1)$. We have $SA(\Phi)S^{-1} = A(\hat{\Phi})$ where $S = \mathrm{diag}(\xi_{11}, \ldots, \xi_{1m_1}, \ldots, \xi_{p1}, \ldots, \xi_{pm_p})$. It means that $\Phi \in [\hat{\Phi}]$. Let Φ' be the subgraph of $\hat{\Phi}$ induced by $\{v_1^1, v_2^1, \ldots, v_p^1\}$, that is, $\Phi' = \hat{\Phi}[v_1^1, v_2^1, \ldots, v_p^1]$. It is clear that Φ' is reduced and $\hat{\Phi} = \Phi' \circ \mathbf{m}$ where $\mathbf{m} = (m_1, m_2, \ldots, m_p)$. The result follows. ■

Remark 2.1: The $\mathbb{T}_4$-gain graph Φ' constructed in Lemma 2.1 is called a *twin reduction graph* of Φ. Note that there may be many twin reduction graphs of Φ but they are switching equivalent.

The characterization of simple graphs with exactly one positive eigenvalue has been given by Smith [9].

Lemma 2.2 ([9]): *Let G be a simple graph. Then $p(G) = 1$ if and only if its non-isolated vertices induce a complete multipartite graph.*

Next, we present the following elementary theorem. Let $C : v_1v_2 \dots v_nv_1$ be a cycle of $\Phi = (G, \mathbb{T}_4, \varphi)$. The *gain* $\varphi(C)$ of C is the product $\varphi(v_1v_2)\varphi(v_2v_3)\cdots\varphi(v_nv_1)$. The cycle C is *balanced* if $\varphi(C) = 1$. A $\mathbb{T}_4$-gain graph Φ is said to be *balanced* if no unbalanced cycles exist, otherwise it is called *unbalanced*.

Lemma 2.3 ([13]): *For a $\mathbb{T}_4$-gain graph $\Phi = (G, \mathbb{T}_4, \varphi)$, it is balanced if and only if every induced cycle of Φ is balanced if and only if $\Phi \in [G]$.*

Now we present some known results on inertia indices of graphs.

Lemma 2.4 ([12]): *Let $\Phi = (G, \mathbb{T}_4, \varphi)$ be a $\mathbb{T}_4$-gain graph with $u \in V(G)$. Then $p(\Phi) - 1 \le p(\Phi - u) \le p(\Phi)$, $n(\Phi) - 1 \le n(\Phi - u) \le n(\Phi)$.*

Lemma 2.5 ([14]): *Let $\Phi = (G, \mathbb{T}_4, \varphi)$ be a $\mathbb{T}_4$-gain graph containing a pendant vertex v with the unique neighbour u. Then $p(\Phi) = p(\Phi - u - v) + 1$, $n(\Phi) = n(\Phi - u - v) + 1$, $\eta(\Phi) = \eta(\Phi - u - v)$. Moreover,* $\mathrm{rank}(\Phi) = \mathrm{rank}(\Phi - u - v) + 2$.

We end up this part by some known results on gain graphs with exactly one positive eigenvalue, which are borrowed from [11] and will be used later.

Lemma 2.6 ([11]): *Let $\Phi = (G, \mathbb{T}_4, \varphi)$ be a $\mathbb{T}_4$-gain graph and u be a vertex of G. If $N_G(u) = \{v_1, \dots, v_s\}$, then there exists $\Phi' = (G, \mathbb{T}_4, \varphi') \in [\Phi]$ such that $\varphi'(uv_i) = 1$ for any $v_i \in N_G(u)$.*

Lemma 2.7 ([11]): *If the connected $\mathbb{T}_4$-gain graph $\Phi = (G, \mathbb{T}_4, \varphi)$ has exactly one positive eigenvalue, then there exists a standard k-partite graph $\Phi' = (K_{n_1,n_2,\dots,n_k}; [\eta_{st} \mid 2 \le s < t \le k])$ such that $\Phi \in [\Phi']$.*

Lemma 2.8 ([11]): *Let $\Phi = (G, \mathbb{T}_4, \varphi)$ be a $\mathbb{T}_4$-gain graph with order n, and H be a subgraph of G. If the eigenvalues of Φ and $\Phi[H]$ are $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$ and $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_m$ respectively, then $\lambda_{n-m+i} \le \mu_i \le \lambda_i$ for $1 \le i \le m$.*

To depict $\mathbb{T}_4$-gain graphs in Figures 1–4, each continuous (respectively, dashed) thick undirected line segment represents two opposite oriented edges with gain 1 (respectively, -1), whereas the arrows detect the oriented edges uv such that $\varphi(uv) = \mathbf{i}$. The other possible choice for the arrow direction not employed here-namely using an arrow from v to u to denote the oriented edge uv such that $\varphi(uv) = \mathbf{i}$-would lead to an alternative and fully satisfactory way to 'read' the imaginary gains from the drawings.

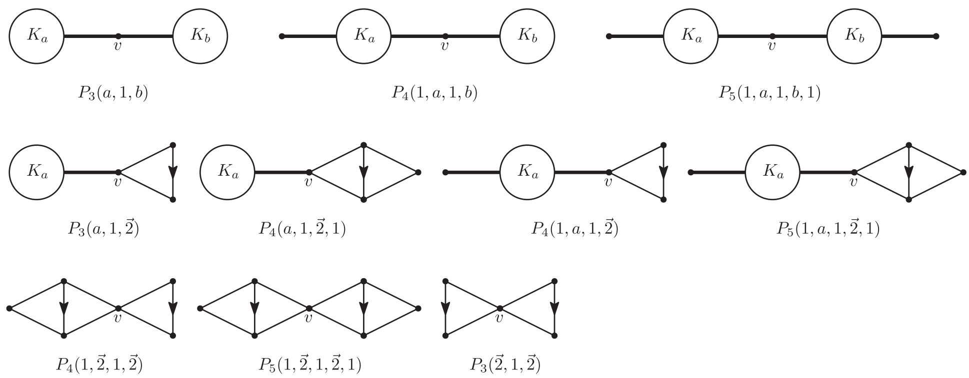

Figure 1. $\mathbb{T}_4$-gain graphs in Theorem 4.4, where $a, b \geq 2$ and a bold line segment connecting two parts represents that each vertex in one part is adjacent to all vertices in the other part.

3. $\mathbb{T}_4$-gain graphs with exactly one positive eigenvalue

Firstly, we consider the $\mathbb{T}_4$-gain graphs with underlying graph being K_4.

Lemma 3.1: *Let* $\Phi = (K_4, \mathbb{T}_4, \varphi)$ *be a* $\mathbb{T}_4$*-gain graph. Then* $\operatorname{rank}(\Phi) = 4$*. If* $\Phi = (K_4, \mathbb{T}_4, \varphi)$ *is unbalanced, then* $p(\Phi) \geq 2$*.*

Proof: Let $V(K_4) = \{v_1, v_2, v_3, v_4\}$. By Lemma 2.6, there exists $\Phi' = (K_4, \mathbb{T}_4, \varphi') \in [\Phi]$ such that $\varphi'(v_1 v_s) = 1$ for any $2 \leq s \leq 4$. The adjacency matrix of Φ' can be written as

$$A(\Phi') = \begin{array}{c c} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & x & y \\ 1 & x^* & 0 & z \\ 1 & y^* & z^* & 0 \end{pmatrix} \end{array}.$$

Then, $\det(A(\Phi')) = 3 - 2\operatorname{Re}(xz + xy^* + yz^*) \neq 0$. Thus, $\operatorname{rank}(\Phi) = \operatorname{rank}(\Phi') = 4$.

If $\Phi = (K_4, \mathbb{T}_4, \varphi)$ is unbalanced, suppose to the contrary that Φ has exactly one positive eigenvalue. Let $\mu_1 \geq \mu_2 \geq \mu_3 \geq \mu_4$ be the eigenvalues of Φ'. For any $2 \leq s < t \leq 4$, the characteristic polynomial of $\Phi'[v_1, v_s, v_t]$ is $f(\lambda) = \lambda^3 - 3\lambda - 2\operatorname{Re}(\varphi'(v_s v_t))$. Since $\mu_2(\Phi'[v_1, v_s, v_t]) \leq 0$ due to $\mu_2(\Phi') \leq 0$ and Lemma 2.8, we get $f(0) = -2\operatorname{Re}(\varphi'(v_s v_t)) \leq 0$ and thus $\operatorname{Re}(\varphi'(v_s v_t)) \geq 0$. Since Φ' is an unbalanced $\mathbb{T}_4$-gain graph and $\operatorname{Re}(x), \operatorname{Re}(y), \operatorname{Re}(z) \geq 0$, we have $x, y, z \neq -1$ and at least one of x, y, z, x^*, y^*, z^* is equal to $\mathbf{i}$. Without loss of generality, assume $x = \mathbf{i}$. In what follows, we show that $\det(A(\Phi')) > 0$.

If $\det(A(\Phi')) = 3 - 2\operatorname{Re}(xz + xy^* + yz^*) \leq 0$, then $\operatorname{Re}(xz + xy^* + yz^*)$ is equal to 2 or 3. If $\operatorname{Re}(xz + xy^* + yz^*) = 2$, then of the three numbers $\operatorname{Re}(xz), \operatorname{Re}(xy^*)$ and $\operatorname{Re}(yz^*)$, exactly one is 0 and the other two are 1. When $\operatorname{Re}(xz) = 0$, we have $\operatorname{Re}(xy^*) = \operatorname{Re}(yz^*) = 1$, and equality holds only if $x = y = z = \mathbf{i}$. Thus $xz = \mathbf{i}^2 = -1$ contradicts $\operatorname{Re}(xz) = 0$. Similarly, $\operatorname{Re}(xy^*) = 0$ or $\operatorname{Re}(yz^*) = 0$ is impossible. Therefore, $\operatorname{Re}(xz + xy^* + yz^*) = 3$, and $\operatorname{Re}(xz) = \operatorname{Re}(xy^*) = \operatorname{Re}(yz^*) = 1$. Since $x = \mathbf{i}$, we have $\operatorname{Re}(xz) = \operatorname{Re}(xy^*) = 1$ holds only if $y = \mathbf{i}, z = -\mathbf{i}$. Thus $yz^* = \mathbf{i}^2 = -1$ contradicts $\operatorname{Re}(yz^*) = 1$. Hence, $\det(A(\Phi')) > 0$. Therefore, Φ' and hence Φ has at least two positive eigenvalues, a contradiction. ■

Now we are ready to present one of our main results.

Theorem 3.1: *Let $\Phi = (G, \mathbb{T}_4, \varphi)$ be a connected $\mathbb{T}_4$-gain graph. Then Φ has exactly one positive eigenvalue if and only if $\Phi \in [K_{n_1,\dots,n_k}]$ for some k or $\Phi \in [\vec{K}_{n_1,n_2,n_3}]$.*

Proof: Note that $\vec{K}_{1,1,1}$ is a twin reduction graph of $\vec{K}_{n_1,n_2,n_3}$. By Lemma 2.1, $p(\vec{K}_{n_1,n_2,n_3}) = p(\vec{K}_{1,1,1})$. The sufficiency follows from $p(\vec{K}_{1,1,1}) = 1$ and Lemma 2.2. In what follows, we show the necessity.

If Φ is balanced, by Lemma 2.3, Φ is switching equivalent to its underlying graph. From Lemma 2.2, we deduce that $\Phi \in [K_{n_1,\dots,n_k}]$ for some k. If Φ is unbalanced, by Lemmas 2.7 and 3.1, G is a complete tripartite graph and there exists a standard 3-partite graph $\Phi' = (K_{n_1,n_2,n_3}; [\eta_{23}])$ such that $\Phi \in [\Phi']$. Note that $(K_{1,1,1}; [\eta_{23}])$ is a twin reduction graph of Φ'. We have $p(K_{1,1,1}; [\eta_{23}]) = p(\Phi') = 1$, which leads to $\eta_{23} \in \{\pm\mathbf{i}\}$ by a direct calculation. Hence, $\Phi \in [\vec{K}_{n_1,n_2,n_3}]$. ■

Remark 3.1: It is well known that the connected graphs with exactly one positive eigenvalue are complete multipartite graphs. It is natural to investigate other graphs with exactly one positive eigenvalue, such as signed graphs, mixed graphs and so on. Since all such graphs are special cases of complex unit gain graphs, in [11] the authors try to completely determine all complex unit gain graphs with exactly one positive eigenvalues. However, it seems impossible to give a complete characterization of such graphs because the gain of each edge could be arbitrary complex unit and the eigenvalues of a complex unit gain graph strongly rely on the gains of it edges. Thus, in [11] the authors just gave a necessary condition on connected complex unit gain graphs with exactly one positive eigenvalue. So, we restrict this problem on $\mathbb{T}_4$-gain graphs, and Theorem 3.1 completely solved it. In fact, Theorem 3.1 is a specialization of the analogous result in [11].

Recall that mixed graphs and signed graph are special $\mathbb{T}_4$-gain graphs. Theorem 3.1 implies one of the results in [8] and one of the results in [7].

Corollary 3.1 ([8]): *Let $\widetilde{G}$ be a connected mixed graph. Then $p(\widetilde{G}) = 1$ if and only if $\Phi \in [K_{n_1,\dots,n_k}]$ for some k or $\Phi \in [\vec{K}_{n_1,n_2,n_3}]$.*

Corollary 3.2 ([7]): *Let $\Gamma(G)$ be a connected signed graph. Then $p(\Gamma(G)) = 1$ if and only if $\Gamma(G) \in [K_{n_1,\dots,n_k}]$ for some k.*

4. On $\mathbb{T}_4$-gain graphs with exactly two positive eigenvalues

We first consider $\mathbb{T}_4$-gain graphs with pendant vertices.

Theorem 4.1: *Let $\Phi = (G, \mathbb{T}_4, \varphi)$ be a connected $\mathbb{T}_4$-gain graph with pendant vertices. Then $p(\Phi) = 2$ if and only if Φ is obtained by adding some $\mathbb{T}_4$-gain edges between the centre of a $\mathbb{T}_4$-gain star and some vertices of F^{φ}, where $F^{\varphi} \in [K_{n_1,\dots,n_k}]$ for some k or $F^{\varphi} \in [\vec{K}_{n_1,n_2,n_3}]$.*

Proof: The sufficiency is immediately from Theorem 3.1 and Lemma 2.5, and we prove the necessity in what follows.

Assume that Φ is a $\mathbb{T}_4$-gain graph and v_1 is a pendant vertex of Φ with neighbour v_2. Since $p(\Phi) = 2$, by Lemma 2.5, we have $p(\Phi - v_1 - v_2) = p(\Phi) - 1 = 1$. Let F^{φ} be the subgraph induced by all non-isolated vertices of $\Phi - v_1 - v_2$. Therefore, Theorem 3.1 implies that $F^{\varphi} \in [K_{n_1,\ldots,n_k}]$ for some k or $F^{\varphi} \in [\vec{K}_{n_1,n_2,n_3}]$. The result follows. ■

Lemma 2.1 indicates that in order to characterize all $\mathbb{T}_4$-gain graphs having exactly two positive eigenvalues, it suffices to characterize all reduced ones having exactly two positive eigenvalues. Since the $\mathbb{T}_4$-gain graphs having pendant vertices with $p(\Phi) = 2$ have been characterized in Theorem 4.1, it only needs to consider the reduced $\mathbb{T}_4$-gain graphs without pendant vertices. In what follows, all graphs are assumed to be reduced graphs without pendant vertices if there is no additional statement.

For convenience, denote by $\mathcal{G}$ the set of connected reduced $\mathbb{T}_4$-gain graphs with a cut vertex, which have no pendant vertices and have exactly two positive eigenvalues. Now, our goal is to determine $\mathcal{G}$. Let $\Phi = (G, \mathbb{T}_4, \varphi) \in \mathcal{G}$ be an arbitrary $\mathbb{T}_4$-gain graph with cut vertex v. Therefore, our goal is to determine Φ.

Lemma 4.1: *The graph $\Phi - v$ has exactly two components $\Phi_1 = (G_1, \mathbb{T}_4, \varphi_1)$ and $\Phi_2 = (G_2, \mathbb{T}_4, \varphi_2)$, and $p(\Phi_i) = 1$ for $1 \le i \le 2$.*

Proof: Suppose that $\Phi - v = \Phi_1 \bigcup \Phi_2 \bigcup \cdots \bigcup \Phi_t$ is the disjoint union of different components of $\Phi - v$. Since v is a cut vertex of Φ, we have $t \ge 2$. Since Φ has no pendant vertices, we have Φ_i has at least one edge for all $i = 1, 2, \ldots, t$. Hence, $p(\Phi_i) \ge 1$ for all $i = 1, 2, \ldots, t$. From $2 = p(\Phi) \ge p(\Phi - v) = p(\Phi_1) + \cdots + p(\Phi_t)$, it follows that $t = 2$. Hence, $\Phi - v$ has exactly two components Φ_1 and Φ_2 and $p(\Phi_i) = 1$ for $1 \le i \le 2$. ■

Lemma 4.2: $\mathrm{rank}(\Phi) \le$ *$\mathrm{rank}(\Phi_1)$* $+ \mathrm{rank}(\Phi_2) + 1$, *and $\mathrm{rank}(\Phi_i + v) \le \mathrm{rank}(\Phi_i) + 1$ for* $1 \le i \le 2$.

Proof: By Lemma 2.4, we have

$$\begin{aligned}\mathrm{rank}(\Phi) &= p(\Phi) + n(\Phi) \le p(\Phi - v) + 1 + n(\Phi - v) + 1\\ &= p(\Phi_1) + p(\Phi_2) + n(\Phi_1) + n(\Phi_2) + 2\\ &= \mathrm{rank}(\Phi_1) + \mathrm{rank}(\Phi_2) + 2.\end{aligned}$$

If the equality holds, then $p(\Phi) = p(\Phi - v) + 1 = p(\Phi_1) + p(\Phi_2) + 1 = 3$, a contradiction. Thus, $\mathrm{rank}(\Phi) \le \mathrm{rank}(\Phi_1) + \mathrm{rank}(\Phi_2) + 1$.

Note that v is a cut vertex of Φ. If $\mathrm{rank}(\Phi_j + v) = \mathrm{rank}(\Phi_j) + 2$ for some j, then $\mathrm{rank}(\Phi) = \mathrm{rank}(\Phi - v) + 2 = \mathrm{rank}(\Phi_1) + \mathrm{rank}(\Phi_2) + 2$ contradicts the first statement. Hence, $\mathrm{rank}(\Phi_i + v) \le \mathrm{rank}(\Phi_i) + 1$ for $1 \le i \le 2$. ■

Clearly, $1 \le p(\Phi_i + v) \le 2$ for $1 \le i \le 2$. Without loss of generality, assume that $p(\Phi_1 + v) \le p(\Phi_2 + v)$. Combining Lemmas 4.1 and 4.2, we get the following result.

Lemma 4.3: $p(\Phi_1 + v) = 1$.

Proof: Suppose to the contrary that $p(\Phi_i + v) = 2$ for $1 \le i \le 2$. The adjacency matrix of Φ is

$$A(\Phi) = \begin{pmatrix} A(\Phi_1) & \tau & 0 \\ \tau^* & 0 & \psi^* \\ 0 & \psi & A(\Phi_2) \end{pmatrix}.$$

By Lemma 4.2, both equations $A(\Phi_1)X = \tau$ and $A(\Phi_2)Y = \psi$ have solutions. Let X_1 (resp. Y_1) be a solution to $A(\Phi_1)X = \tau$ (resp. $A(\Phi_2)Y = \psi$). Set

$$W = \begin{pmatrix} I & -X_1 & 0 \\ 0 & 1 & 0 \\ 0 & -Y_1 & I \end{pmatrix}, \quad \text{and thereby}$$

$$W^*A(\Phi)W = \begin{pmatrix} A(\Phi_1) & 0 & 0 \\ 0 & -\tau^*X_1 - \psi^*Y_1 & 0 \\ 0 & 0 & A(\Phi_2) \end{pmatrix}.$$

Since $p(W^*A(\Phi)W) = p(\Phi) = p(\Phi_1) + p(\Phi_2)$, we have $-\tau^*X_1 - \psi^*Y_1 \le 0$. From $p(\Phi_1 + v) = p(\Phi_2 + v) = 2$, we have $-\tau^*X_1 > 0$ and $-\psi^*Y_1 > 0$. Hence, $-\tau^*X_1 - \psi^*Y_1 > 0$, a contradiction. ■

Next, we divide two cases to discuss, namely, $p(\Phi_i + v) = 1$ for $1 \le i \le 2$, and $p(\Phi_1 + v) = 1$ and $p(\Phi_2 + v) = 2$.

Theorem 4.2: *If $p(\Phi_i + v) = 1$ for $1 \le i \le 2$, then Φ is switching equivalent to one of the following $\mathbb{T}_4$-gain graphs present in Figure 1:*

(i) $P_3(a, 1, b)$, $P_4(1, a, 1, b)$, $P_5(1, a, 1, b, 1)$, *where* $a, b \ge 2$;
(ii) $P_3(a, 1, \vec{2})$, $P_4(a, 1, \vec{2}, 1)$, $P_4(1, a, 1, \vec{2})$, $P_5(1, a, 1, \vec{2}, 1)$, $P_4(1, \vec{2}, 1, \vec{2})$, $P_5(1, \vec{2}, 1, \vec{2}, 1)$, $P_3(\vec{2}, 1, \vec{2})$.

Proof: For $1 \le i \le 2$, since $p(\Phi_i + v) = 1$, Theorem 3.1 indicates that $\Phi_i + v \in [K_{n_1,\ldots,n_k}]$ for some k or $\Phi_i + v \in [\vec{K}_{n_1,n_2,n_3}]$. In what follows, we divide three cases to discuss.

Case 1. $\Phi_1 + v \in [K_{l_1,\ldots,l_s}]$ and $\Phi_2 + v \in [K_{n_1,\ldots,n_t}]$.

Without loss of generality, assume that $l_1 \ge l_2 \ge \cdots \ge l_s$ and $n_1 \ge n_2 \ge \cdots \ge n_t$. Since Φ is reduced, we have $l_i = n_j = 1$ for $2 \le i \le s$ and $2 \le j \le t$, and $1 \le l_1, n_1 \le 2$. Furthermore, if $l_1 = 2$ (or $n_1 = 2$), then there is a vertex $v' \in \Phi_1$ (resp. $v' \in \Phi_2$) such that v and v' are twins in $\Phi_1 + v$ (resp. $\Phi_2 + v$). If $l_1 = n_1 = 1$, then $\Phi \in [P_3(s-1, 1, t-1)]$. If one of l_1 and n_1 equal to 1, say $l_1 = 1$ and $n_1 = 2$, then $\Phi \in [P_4(s-1, 1, t-1, 1)]$. If $l_1 = n_1 = 2$, then $\Phi \in [P_5(1, s-1, 1, t-1, 1)]$.

Case 2. $\Phi_1 + v \in [\vec{K}_{l_1,l_2,l_3}]$ and $\Phi_2 + v \in [\vec{K}_{n_1,n_2,n_3}]$.

Without loss of generality, assume that $l_1 \ge l_2 \ge l_3$ and $n_1 \ge n_2 \ge n_3$. Since Φ is reduced, we have $l_i = n_j = 1$ for $2 \le i, j \le 3$, and $1 \le l_1, n_1 \le 2$. Furthermore, if $l_1 = 2$ (or $n_1 = 2$), then there is a vertex $v' \in \Phi_1$ (resp. $v' \in \Phi_2$) such that v and v' are twins in $\Phi_1 + v$ (resp. $\Phi_2 + v$). If $l_1 = n_1 = 1$, then $\Phi \in [P_3(\vec{2}, 1, \vec{2})]$. If one of l_1 and n_1 equal to 1, say $l_1 = 1$ and $n_1 = 2$, then $\Phi \in [P_4(\vec{2}, 1, \vec{2}, 1)]$. If $l_1 = n_1 = 2$, then $\Phi \in [P_5(1, \vec{2}, 1, \vec{2}, 1)]$.

Case 3. $\Phi_1 + v \in [K_{l_1,\ldots,l_s}]$ and $\Phi_2 + v \in [\vec{K}_{n_1,n_2,n_3}]$.

Similarly, one could easily verify that $\Phi \in [P_3(s-1,1,\vec{2})] \cup [P_4(1,s-1,1,\vec{2})] \cup [P_4(s-1,1,\vec{2},1)] \cup [P_5(1,s-1,1,\vec{2},1)]$. ■

It remains to consider the case that $p(\Phi_1+v)=1$ and $p(\Phi_2+v)=2$.

Lemma 4.4: *If $p(\Phi_1+v)=1$ and $p(\Phi_2+v)=2$, then we have*

(i) $\Phi_1+v \in [K_q]$ *for some* q *or* $\Phi_1+v \in [\vec{K}_{1,1,1}]$*;*
(ii) $\Phi_2 \in [K_t]$ *for some* t *or* $\Phi_2 \in [\vec{K}_{1,1,1}]$.

Proof: By Theorem 3.1, Φ_1+v is a balanced complete multipartite graphs or switching equivalent to $\vec{K}_{n_1,n_2,n_3}$. Suppose to the contrary that (i) is not true. Since Φ is reduced, we conclude that there exists a vertex v_1 in Φ_1 which is a twin point of v in Φ_1+v. Therefore, $vv_1 \notin E(G)$ and $N_{G_1+v}(v_1)=N_{G_1+v}(v)$. Let $u \in N_{G_1+v}(v_1)$. Since v_1u is a pendant edge of $\Phi[\{v_1,u,v\}\cup V(G_2)]$, by Lemma 2.5, we have

$$p(\Phi) \ge p(\Phi[\{v_1,u,v\}\cup V(G_2)]) = p(\Phi_2+v)+1 = 3,$$

a contradiction. Thus, (i) holds.

By Lemma 4.1 and Theorem 3.1, Φ_2 is a balanced complete multipartite graph or switching equivalent to $\vec{K}_{n_1,n_2,n_3}$. Suppose for a contradiction that (ii) is not true. Therefore, Φ_2 has two vertices u_1,u_2 such that $N_{G_2}(u_1)=N_{G_2}(u_2)$ and $\varphi(u_1w)=\xi\varphi(u_2w)$ for all $w \in N_{G_2}(u_1)$, where $\xi \in \{1,-1,\mathbf{i},-\mathbf{i}\}$. Since Φ is reduced, we know at least one of u_1,u_2, say u_1, is adjacent to v.

If u_2 is not adjacent to v, then

$$N_G(u_1)=N_{G_2}(u_1)\cup\{v\}=N_{G_2}(u_2)\cup\{v\}=N_G(u_2)\cup\{v\}.$$

Let $\widetilde{\Phi}$ be a $\mathbb{T}_4$-gain graph obtained from Φ by deleting all edges $\{u_1,w\}$ for $w \in N_{G_2}(u_1)$. Let E_{u_1,u_2} be the matrix of order $|V(G)|$ whose unique nonzero entry 1 is at the row indexed by u_1 and the column indexed by u_2. Since

$$A(\widetilde{\Phi}) = (I_{|V(G)|}-\xi E_{u_1,u_2})A(\Phi)(I_{|V(G)|}-\xi E_{u_1,u_2})^*,$$

we have $p(\Phi)=p(\widetilde{\Phi})$. Then, u_1 is a pendant vertex adjacent to v in $\widetilde{\Phi}$. Using Lemma 2.5, we have

$$p(\Phi)=p(\widetilde{\Phi})=p(\widetilde{\Phi}-u_1-v)+1=p(\Phi_1)+p(\Phi_2-u_1)+1=3,$$

a contradiction.

If u_2 is adjacent to v, then $\varphi(u_1v)\neq\xi\varphi(u_2v)$ because Φ is reduced. Therefore, from (i), Φ must contain an induced subgraph switching equivalent to one of the $\mathbb{T}_4$-gain graphs F_1,F_2,F_3 and F_4 depicted in Figure 2. By calculations, for $1\le j\le 4$, we have

$$\det(A(F_j)) = 2\mathrm{Re}(\varphi_j(vu_1)\varphi_j(vu_2)^*)-2 \le 0,$$

with equality if and only if $\varphi_j(vu_1)=\varphi_j(vu_2)$. Since

$$\varphi_j(vu_1)=\varphi_j(u_1v)^* \neq \varphi_j(u_2v)^* = \varphi_j(vu_2),$$

we have $\det(A(F_j))\neq 0$ and F_1,F_2,F_3,F_4 all have at least three positive eigenvalues. Hence, by Lemma 2.8, we obtain $p(\Phi)\ge 3$, a contradiction. Thus, (ii) holds. ■

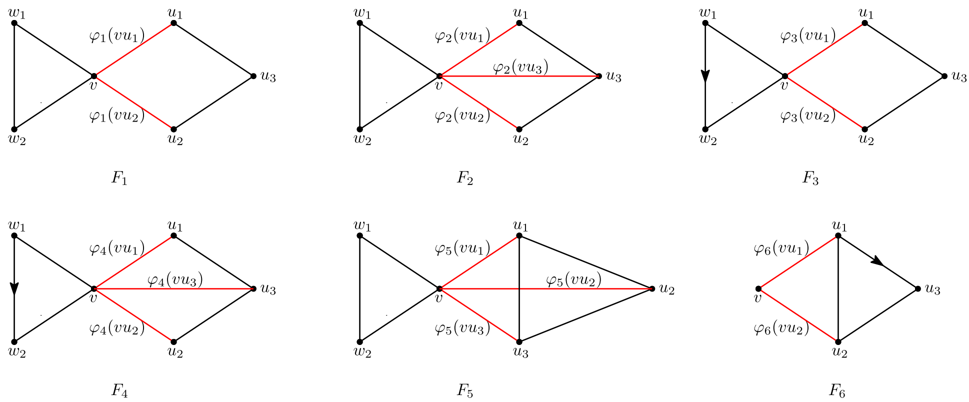

Figure 2. $\mathbb{T}_4$-gain graphs $F_1, F_2, \ldots, F_6$, where $\varphi_j(vu_1) \neq \varphi_j(vu_2)$ $(1 \leq j \leq 4)$, and $\varphi_5(vu_1), \varphi_5(vu_2), \varphi_5(vu_3)$ are different from each other.

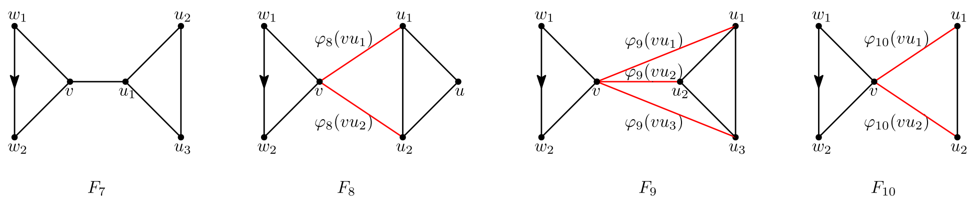

Figure 3. $\mathbb{T}_4$-gain graphs F_7, F_8, F_9, F_{10}, where all unmarked edges are assigned 1.

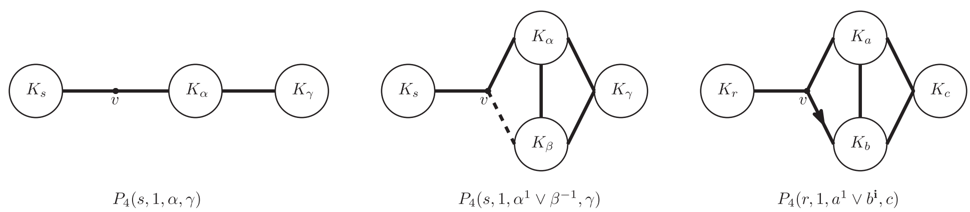

Figure 4. $\mathbb{T}_4$-gain graphs $P_4(s, 1, \alpha, \gamma), P_4(s, 1, \alpha^1 \vee \beta^{-1}, \gamma)$ and $P_4(r, 1, a^1 \vee b^{\mathbf{i}}, c)$.

Lemma 4.4 indicates the possible structures of $\Phi_1 + v$ and Φ_2. In fact, we further get the following results.

Lemma 4.5: *If $p(\Phi_1 + v) = 1$ and $p(\Phi_2 + v) = 2$, then $\Phi_2 \in [K_t]$ for some t.*

Proof: Lemma 4.4 indicates that either $\Phi_2 \in [K_t]$ or $\Phi_2 \in [\vec{K}_{1,1,1}]$. Suppose to the contrary that $\Phi_2 = [\vec{K}_{1,1,1}]$. If v is adjacent to exactly one vertex of Φ_2, let u be the vertex of Φ_2 adjacent to v, then Lemma 2.5 implies that $\text{rank}(\Phi_2 + v) = \text{rank}(\Phi_2 - u) + 2 = 4 = \text{rank}(\Phi_2) + 2$, contradicts Lemma 4.2; if v is adjacent to all vertices of Φ_2 in $\Phi_2 + v$, then Lemma 3.1 implies that $\text{rank}(\Phi_2 + v) = 4 = \text{rank}(\Phi_2) + 2$, contradicts Lemma 4.2. Therefore, v is adjacent to exactly two vertices of Φ_2 in $\Phi_2 + v$, and thus $\Phi_2 + v$ is switching equivalent to the form of F_6 (see Figure 2). The adjacency matrix of F_6 is

$$A(F_6) = \begin{pmatrix} 0 & \varphi_6(vu_1) & \varphi_6(vu_2) & 0 \\ \varphi_6(vu_1)^* & 0 & 1 & \mathbf{i} \\ \varphi_6(vu_2)^* & 1 & 0 & 1 \\ 0 & -\mathbf{i} & 1 & 0 \end{pmatrix}.$$

By immediate calculations, the characteristic polynomial of F_6 is equal to

$$f(\lambda) = \lambda^4 - 5\lambda^2 - \mathrm{Re}(\varphi_6(vu_1)\varphi_6(vu_2)^*)\lambda + 2[1 + \mathrm{Im}(\varphi_6(vu_1)\varphi_6(vu_2)^*)].$$

Since $p(\Phi_2) = 1$ and $p(\Phi_2 + v) = 2$, we have $\mathrm{rank}(\Phi_2 + v) \geq \mathrm{rank}(\Phi_2) + 1$. Note that Lemma 4.2 indicates that $\mathrm{rank}(\Phi_2 + v) \leq \mathrm{rank}(\Phi_2) + 1$. We have $\mathrm{rank}(F_6) = \mathrm{rank}(\Phi_2 + v) = \mathrm{rank}(\Phi_2) + 1 = 3$, which leads to that the coefficient of λ is not equal to 0 and $f(\lambda) = 0$. It leads to $\mathrm{Re}(\varphi_6(vu_1)\varphi_6(vu_2)^*) \neq 0$ and $1 + \mathrm{Im}(\varphi_6(vu_1)\varphi_6(vu_2)^*) = 0$, which cannot hold at the same time. ■

Lemma 4.6: *If $p(\Phi_1 + v) = 1$ and $p(\Phi_2 + v) = 2$, then $\Phi_1 + v \in [K_s]$ for some s.*

Proof: Suppose to the contrary that $\Phi_1 + v \in [\vec{K}_{1,1,1}]$. Lemma 4.5 indicates that $\Phi_2 \in [K_t]$ for some t. If there are two vertices of Φ_2 not adjacent to v in Φ, then Φ has an induced subgraph switching equivalent to F_7. By direct calculations, we have $p(F_7) = 3$, and thus $p(\Phi) \geq 3$ according to Lemma 2.8, a contradiction. If there is exactly one vertex u of Φ_2 not adjacent to v in Φ, then Φ has an induced subgraph switching equivalent to F_8. Lemma 2.8 implies that $2 = p(F_8 - v) \leq p(F_8) \leq p(\Phi) = 2$, and thereby $p(F_8) = 2$. By direct calculations, $p(F_8) = 2$ if and only if $\varphi_8(vu_1) = \varphi_8(vu_2)$. By the arbitrariness of u_1 and u_2, all edges from v to Φ_2 obtain the same gain in $\mathbb{T}_4$. Hence, v and u are twin points in $\Phi_2 + v$. And $p(\Phi_2 + v) = p(\Phi_2) = 1$, a contradiction. Therefore, v is adjacent to all vertices of Φ_2, and thus Φ is switching equivalent to a $\mathbb{T}_4$-gain graph $\widehat{\Phi}$ such that $\widehat{\Phi}[V(\Phi_1) \cup \{v\}] = \vec{K}_{1,1,1}$ and $\widehat{\Phi}[V(\Phi_2)] = K_t$. Next we prove $t = 2$. Otherwise, let u_1, u_2, u_3 be three vertices of $\widehat{\Phi}[V(G_2)]$. Then $\widehat{\Phi}[V(G_1) \cup \{v, u_1, u_2, u_3\}]$ is F_9. By Lemma 2.8, we have $2 = p(F_9 - v) \leq p(F_9) \leq 2$. Hence, $p(F_9) = 2$. Since it is easy to see that $F_9 - v$ has three negative eigenvalues, Lemma 2.8 implies that F_9 has at least three negative eigenvalues. It means that $p(F_9) = 2$ if and only if $\det(A(F_9)) \geq 0$. By immediate calculations, we have

$$\begin{aligned} &\det(A(F_9)) \\ &= 2\left[\mathrm{Re}(\varphi_9(vu_1)\varphi_9(vu_2)^*) + \mathrm{Re}(\varphi_9(vu_1)\varphi_9(vu_3)^*) + \mathrm{Re}(\varphi_9(vu_2)\varphi_9(vu_3)^*)\right] - 3. \end{aligned}$$

It is easy to see that $\det(A(F_9)) \geq 0$ if and only if $\varphi_9(vu_1) = \varphi_9(vu_2) = \varphi_9(vu_3)$. By the arbitrariness of u_1, u_2, u_3, all the edges from v to $V(G_2)$ in $\widehat{\Phi}$ obtain the same gain. Hence, $\widehat{\Phi}$ is switching equivalent to a $\mathbb{T}_4$-gain graph $\widehat{\Phi}_1$ such that $\widehat{\Phi}_1[V(G_2) \cup \{v\}] \in [K_{t+1}]$. Thus, $\Phi_2 + v \in [K_{t+1}]$, contradicts the assumption of $p(\Phi_2 + v) = 2$. Hence, $t = 2$.

By the above discussions, we have $\Phi \in [F_{10}]$. Hence, $\Phi_2 + v$ is switching equivalent to $F_{10}[v, u_1, u_2]$, and thereby $p(F_{10}[v, u_1, u_2]) = 2$. It yields, by direct calculations, that $\varphi_{10}(vu_1) = -\varphi_{10}(vu_2)$. Note that we also have $p(\Phi) = p(F_{10}) = 2$. By immediate calculations, we have either $\varphi_{10}(vu_1) = \varphi_{10}(vu_2)$ or $\varphi_{10}(vu_1) = (\pm\mathbf{i})\varphi_{10}(vu_2)$, contradicting the equation $\varphi_{10}(vu_1) = -\varphi_{10}(vu_2)$ (Figure 3).

The proof is completed. ■

Combining Lemmas 4.5 and 4.6, it remains to consider the condition that $\Phi_1 + v \in [K_s]$ for some s and $\Phi_2 \in [K_t]$ for some t. Before discussing this condition, we need to introduce two classes of $\mathbb{T}_4$-gain graphs. For integers $s \geq 2, \alpha \geq \beta \geq 0, \gamma \geq 0$, let $P_4(s, 1, \alpha^1 \vee \beta^{-1}, \gamma)$ be the $\mathbb{T}_4$-gain graph obtained from K_s and $K_{\alpha+\beta+\gamma}$ by adding a new vertex v and adding $s + \alpha + \beta$ edges such that v is adjacent to s vertices of K_s and adjacent to $\alpha + \beta$ vertices of $K_{\alpha+\beta+\gamma}$, where β edges from v to $K_{\alpha+\beta+\gamma}$ obtain gain -1 and all the other edges have gain 1 (see Figure 4). Especially, if $\beta = 0$, we use the notation $P_4(s, 1, \alpha, \gamma)$ for $P_4(s, 1, \alpha^1 \vee 0^{-1}, \gamma)$. For integers $r \geq 2, a \geq b \geq 1, c \geq 0$, let $P_4(r, 1, a^1 \vee b^{\mathbf{i}}, c)$ be the $\mathbb{T}_4$-gain graph obtained from K_r and K_{a+b+c} by adding a new vertex v and adding $r + a + b$ edges such that v is adjacent to r vertices of K_r and adjacent to $a + b$ vertices of K_{a+b+c}, where b edges from v to K_{a+b+c} obtain gain $\mathbf{i}$ and all the other edges have gain 1 (see Figure 4).

Theorem 4.3: *If $p(\Phi_1 + v) = 1$ and $p(\Phi_2 + v) = 2$, then Φ is switching equivalent to one of the following $\mathbb{T}_4$-gain graphs:*

(i) $P_4(s, 1, \alpha, \gamma)$, *where* $s \geq 2, \alpha \geq 1, \gamma \geq 2$, $\frac{1}{\gamma-1} \geq 1 - \frac{1}{s} - \frac{1}{\alpha}$.

(ii) $P_4(2, 1, 1^1 \vee 1^{-1}, \gamma)$, *where* $\gamma \geq 0$.

(iii) $P_4(r, 1, a^1 \vee b^{\mathbf{i}}, c)$, *where* $(a, b, c, r) \in \{(a, 1, c, r) \mid r \geq 2, a \geq 1, \frac{ac-1}{a+c} \leq \frac{1}{r-1}\} \cup \mathcal{S}$ *and*

$$\mathcal{S} = \{(2,2,0,2), (2,2,0,3), (2,2,0,4), (3,2,0,2), (4,2,0,2), (2,2,1,2), (2,2,1,3), (2,2,2,2), (2,2,3,2)\}.$$

Proof: According to Lemmas 4.5 and 4.6, we conclude that $\Phi_1 + v \in [K_s]$ for some s and $\Phi_2 \in [K_t]$ for some t. We claim that there are not three edges between v and Φ_2 that obtain three different gains. Otherwise, Φ will contain an induced subgraph in $[F_5]$, where F_5 is depicted in Figure 2. By a direct calculation, we have $\det(A(F_5)) = -1$, which implies that $p(F_5) \geq 3$, a contradiction. Hence, Φ is switching equivalent to

$$P_4(s, 1, \alpha, \gamma), P_4(s, 1, \alpha^1 \vee \beta^{-1}, \gamma) \quad \text{or} \quad P_4(r, 1, a^1 \vee b^{\mathbf{i}}, c)$$

for some $s, r \geq 2, \alpha \geq \beta \geq 1, a \geq b \geq 1$ and $\gamma, c \geq 0$.

Case 1. $\Phi \in [P_4(s, 1, \alpha, \gamma)]$, where $s \geq 2, \alpha \geq 1$ and $\gamma \geq 0$.

Since $p(\Phi_2 + v) = 2$, we have $\gamma \geq 2$. By the knowledge of equitable partition ([15, Page 198]), the eigenvalues of $P_4(s, 1, \alpha, \gamma)$ consist of -1 with multiplicity $s + \alpha + \gamma - 3$ and the roots of $f(\lambda) = \det(\lambda I - A_\pi)$, where

$$A_\pi = \begin{pmatrix} s-1 & 1 & 0 & 0 \\ s & 0 & \alpha & 0 \\ 0 & 1 & \alpha - 1 & \gamma \\ 0 & 0 & \alpha & \gamma - 1 \end{pmatrix}.$$

It is easy to see that A_π has at least one negative eigenvalue. Therefore, $p(A_\pi) = 2$ if and only if $\det(A_\pi) \geq 0$. By simple calculations, we have

$$\det(A_\pi) = (\alpha + s)(\gamma - 1) - \alpha s(\gamma - 2).$$

Thus $\det(A_\pi) \geq 0$ if and only if $\frac{1}{\gamma-1} \geq 1 - \frac{1}{s} - \frac{1}{\alpha}$.

Case 2. $\Phi \in [P_4(s, 1, \alpha^1 \vee \beta^{-1}, \gamma)]$, where $s \geq 2, \alpha \geq \beta \geq 1$ and $\gamma \geq 0$.

Firstly, by an elementary calculation, we have

$$p(P_4(2, 1, 2^1 \vee 1^{-1}, 0)) = p(P_4(3, 1, 1^1 \vee 1^{-1}, 0)) = 3.$$

Hence, by Lemma 2.8, we have $p(P_4(s, 1, \alpha^1 \vee \beta^{-1}, \gamma)) = 2$ only if $\alpha = \beta = 1$ and $s = 2$.

If $\gamma = 0$, we have $p(P_4(2, 1, 1^1 \vee 1^{-1}, 0)) = 2$. We now consider $\gamma \geq 1$. By the knowledge of equitable partition ([15, Page 198]), the eigenvalues of $P_4(2, 1, 1^1 \vee 1^{-1}, \gamma)$ consist of -1 with multiplicity γ and the roots of

$$\begin{aligned} f(\lambda) = \det(\lambda I - A_\pi) &= \lambda^5 - \gamma\lambda^4 - (6+\gamma)\lambda^3 + 5\gamma\lambda^2 + (5\gamma+5)\lambda \\ &= \lambda(\lambda+1)(\lambda^2 - 5)(\lambda - \gamma - 1) \end{aligned}$$

where

$$A_\pi = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & \gamma \\ 0 & -1 & 1 & 0 & \gamma \\ 0 & 0 & 1 & 1 & \gamma - 1 \end{pmatrix}.$$

It is easy to see that A_π has exactly two positive eigenvalues. Hence, $p(P_4(2, 1, 1^1 \vee 1^{-1}, \gamma)) = p(A_\pi) = 2$.

Case 3. $\Phi \in [P_4(r, 1, a^1 \vee b^{\mathbf{i}}, c)]$, where $r \geq 2, a \geq b \geq 1$ and $c \geq 0$.

Similarly, by using the knowledge of equitable partition, the eigenvalues of $P_4(r, 1, a^1 \vee b^{\mathbf{i}}, c)$ consist of -1 with multiplicity $r + a + b + c - 4$ and the roots of $f(\lambda) = \det(\lambda I - A_\pi)$ where

$$A_\pi = \begin{pmatrix} r-1 & 1 & 0 & 0 & 0 \\ r & 0 & a & b\mathbf{i} & 0 \\ 0 & 1 & a-1 & b & c \\ 0 & -\mathbf{i} & a & b-1 & c \\ 0 & 0 & a & b & c-1 \end{pmatrix},$$

and thus $p(A_\pi) = p(P_4(r, 1, a^1 \vee b^{\mathbf{i}}, c)) = 2$. By a direct calculation, we obtain

$$\det(A_\pi) = a(2b + c)(r - 1) - 2r(a + b) - c(b + r - br) + a + b + r.$$

Note that the matrix

$$B = \begin{pmatrix} a-1 & b & c \\ a & b-1 & c \\ a & b & c-1 \end{pmatrix}$$

is a principal matrix of A_π and the spectrum of B is $\{a + b + c - 1, -1^{[2]}\}$. Hence, by Lemma 2.8, A_π has at least two negative eigenvalues. Therefore, $p(A_\pi) = 2$ if and only if $\det(A_\pi) \leq 0$.

It is easy to see that

$$\det(A_\pi) = \begin{cases} acr - ac - a - c - r + 1, & \text{if } b = 1, \\ a(2r + cr - c - 3) - 3r - 2c + cr + 2, & \text{if } b = 2. \end{cases}$$

Thus, if $b = 1$, then $\det(A_\pi) \leq 0$ if and only if $acr - ac - a - c - r + 1 \leq 0$ if and only if $\frac{ac-1}{a+c} \leq \frac{1}{r-1}$. In what follows, we assume $b = 2$.

- For $c = 0$. If $a \geq 5$, then $\det(A_\pi) \geq 5(2r - 3) - 3r + 2 = 7r - 13 > 0$. Hence, $\det(A_\pi) \leq 0$ only if $a \leq 4$. Note that $a \geq b$. If $a = 2$, then $\det(A_\pi) = r - 4 \leq 0$ if and only if $r = 2, 3$, or 4. If $a = 3$, then $\det(A_\pi) = 3r - 7 \leq 0$ if and only if $r = 2$. If $a = 4$, then $\det(A_\pi) = 5r - 10 \leq 0$ if and only if $r = 2$.
- For $c \geq 1$. If $a \geq 3$, then $\det(A_\pi) \geq 3(2r + cr - c - 3) - 3r - 2c + cr + 2 = r(4c + 3) - (5c + 7) \geq 2(4c + 3) - (5c + 7) = 3c - 1 > 0$. Hence, $\det(A_\pi) \leq 0$ only if $a \leq 2$. Note that $a \geq b$. If $a = 2$, then $\det(A_\pi) = 2(2r + cr - c - 3) - 3r - 2c + cr + 2 \leq 0$ if and only if $c = 1$ and either $r = 2$ or $r = 3$, or $c = r = 2$, or $c = 3$ and $r = 2$.

The proof is completed. ■

Combining Theorems 4.2 and 4.3, we could get one of our main result immediately.

Theorem 4.4: *Let $\Phi = (G, \mathbb{T}_4, \varphi)$ be a connected reduced $\mathbb{T}_4$-gain graph with a cut vertex v and without pendant vertices. Then $p(\Phi) = 2$ if and only if Φ is switching equivalent to one of the following $\mathbb{T}_4$-gain graphs:*

(i) $P_3(a, 1, b)$, $P_4(1, a, 1, b)$, $P_5(1, a, 1, b, 1)$, *where* $a, b \geq 2$;
(ii) $P_3(a, 1, \vec{2})$, $P_4(a, 1, \vec{2}, 1)$, $P_4(1, a, 1, \vec{2})$, $P_5(1, a, 1, \vec{2}, 1)$, $P_4(1, \vec{2}, 1, \vec{2})$, $P_5(1, \vec{2}, 1, \vec{2}, 1)$, $P_3(\vec{2}, 1, \vec{2})$;
(iii) $P_4(s, 1, \alpha, \gamma)$, *where* $s \geq 2, \alpha \geq 1, \gamma \geq 2$, $\frac{1}{\gamma-1} \geq 1 - \frac{1}{s} - \frac{1}{\alpha}$;
(iv) $P_4(2, 1, 1^1 \vee 1^{-1}, \gamma)$, *where* $\gamma \geq 0$;
(v) $P_4(r, 1, a^1 \vee b^{\mathbf{i}}, c)$, *where* $(a, b, c, r) \in \{(a, 1, c, r) \mid r \geq 2, a \geq 1, \frac{ac-1}{a+c} \leq \frac{1}{r-1}\} \cup \mathcal{S}$.

Theorem 4.4 immediately implies the following result on mixed graphs.

Corollary 4.1: *Let $\widetilde{G}$ be a connected twin reduction mixed graph with a cut vertex v and without pendant vertices. Then $p(\widetilde{G}) = 2$ if and only if $\widetilde{G}$ is switching equivalent to one of the following graphs:*

(i) $P_3(a, 1, b)$, $P_4(1, a, 1, b)$, $P_5(1, a, 1, b, 1)$, *where* $a, b \geq 2$;
(ii) $P_3(a, 1, \vec{2})$, $P_4(a, 1, \vec{2}, 1)$, $P_4(1, a, 1, \vec{2})$, $P_5(1, a, 1, \vec{2}, 1)$, $P_4(1, \vec{2}, 1, \vec{2})$, $P_5(1, \vec{2}, 1, \vec{2}, 1)$, $P_3(\vec{2}, 1, \vec{2})$;
(iii) $P_4(s, 1, \alpha, \gamma)$, *where* $s \geq 2, \alpha \geq 1, \gamma \geq 2$, $\frac{1}{\gamma-1} \geq 1 - \frac{1}{s} - \frac{1}{\alpha}$;
(iv) $P_4(2, 1, 1^1 \vee 1^{-1}, \gamma)$, *where* $\gamma \geq 0$;
(v) $P_4(r, 1, a^1 \vee b^{\mathbf{i}}, c)$, *where* $(a, b, c, r) \in \{(a, 1, c, r) \mid r \geq 2, a \geq 1, \frac{ac-1}{a+c} \leq \frac{1}{r-1}\} \cup \mathcal{S}$.

Note that $\vec{K}_{1,1,1}$ cannot switching equivalent to a signed graph. We conclude that there are no signed graphs in $[\Phi]$ when Φ is a graph belonging to (ii) or (v) in Theorem 4.4. Therefore, Theorem 4.4 implies the following result on signed graphs.

Corollary 4.2 ([5]): *Let $\Gamma(G)$ be a connected twin reduction signed graph with a cut vertex v and without pendant vertices. Then $p(\Gamma(G)) = 2$ if and only if $\Gamma(G)$ is switching equivalent to one of the following signed graphs:*

(i) $P_3(a, 1, b)$, $P_4(1, a, 1, b)$, $P_5(1, a, 1, b, 1)$, *where* $a, b \geq 2$.
(ii) $P_4(s, 1, \alpha, \gamma)$, *where* $s \geq 2, \alpha \geq 1, \gamma \geq 2$, $\frac{1}{\gamma-1} \geq 1 - \frac{1}{s} - \frac{1}{\alpha}$.
(iii) $P_4(2, 1, 1^1 \vee 1^{-1}, \gamma)$, *where* $\gamma \geq 0$.

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ORCID

Lu Lu http://orcid.org/0000-0003-3138-7546

References

[1] Guo K, Mohar B. Hermitian adjacency matrix of digraphs and mixed graphs. J Graph Theory. 2016;85:217–248.
[2] Liu JX, Li XL. Hermitian-adjacency matrices and hermitian energies of mixed graphs. Linear Algebra Appl. 2015;466:182–207.
[3] Wissing P, van Dam ER. The negative tetrahedron and the first infinite family of connected digraphs that are strongly determined by the hermitian spectrum. J Combin Theory Ser A. 2020;173:105232.
[4] Wei W, Feng ZM, Li SC. Relations between the inertia indices of a mixed graph and those of its underlying graph. Linear Algebra Appl. 2020;588:19–53.
[5] Wang XL, Wong D, Tian FL. Signed graphs with cut points whose positive inertia indexes are two. Linear Algebra Appl. 2018;539:14–27.
[6] Wang Y, Yuan BJ, Li SD, et al. Mixed graphs with H-rank 3. Linear Algebra Appl. 2017;524:22–34.
[7] Yu GH, Feng LH, Qu H. Signed graphs with small positive index of inertia. Electron J Linear Algebra. 2016;31:232–243.
[8] Yuan BJ, Wang Y, Xu J. Characterizing the mixed graphs with exactly one positive eigenvalue and its application to mixed graphs determined by their H-spectra. Appl Math Comput. 2020;380:125279.

[9] Smith JH. Some properties of the spectrum of a graph. In: Combinatorial structures and their applications. New York: Gordon and Breach; 1970. p. 403–406.
[10] Oboudi MR. Characterization of graphs with exactly two non-negative eigenvalues. Ars Math Contemp. 2017;12(2):271–286.
[11] Lu L, Wang JF, Huang QX. Complex unit gain graphs with exactly one positive eigenvalue. Linear Algebra Appl. 2021;608:270–281.
[12] Zaman S, He XC. Relation between the inertia indices of a complex unit gain graph and those of its underlying graph. Linear Multilinear Algebra. 2022;70(5):843–877.
[13] Alazemi A, Anđelić M, Belardo F, et al. Line and subdivision graphs determined by $\mathbb{T}_4$-Gain graphs. Mathematics. 2019;7(10):926. doi:10.3390/math7100926
[14] Yu GH, Qu H, Tu JH. Inertia of complex unit gain graphs. Appl Math Comput. 2015;265:619–629.
[15] Godsil C, Royle G. Algebraic graph theory. New York: Springer; 2001.