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ABSTRACT

In this paper, we consider the spectral radius of signless p-Laplacian of a graph, which is a generalization of the quadratic form of the signless Laplacian matrix for $p = 2$. Let $\mathcal{G}_{n,\beta}$ be the set of simple graphs of order n with a given matching number β. In this paper, the graphs maximizing the largest signless p-Laplacian eigenvalue among $\mathcal{G}_{n,\beta}$ are obtained.

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1. Introduction

We use standard notations from Ref. [1]. Let G be a simple connected graph with vertex set V and edge set E. For two vertices $u, v \in V$, we write $u \sim v$ if $uv \in E$. For a vertex $v \in V$, the *neighbourhood* $N_G(v)$ of v is defined to be $N_G(v) = \{u \in V \mid u \sim v\}$, and the cardinality $d_G(v)$ of $N_G(v)$ is called the degree of v, i.e. $d_G(v) = |N_G(v)|$. As usual, the minimum degree, the maximum degree and the average degree of G are denoted by $\delta(G)$, $\Delta(G)$ and $\overline{d}(G)$, respectively. The *distance* $d_G(u, v)$ of two vertices u and v is the length of the shortest path between u and v.

The Laplacian matrix L of G [2] is defined to be $L = D - A$, where D is the degree diagonal matrix and A is the adjacency matrix. It is well known that L is a linear operator

$$L : \mathbb{R}^V \to \mathbb{R}^V, \quad (Lf)(v) = \sum_{u\sim v}(f(v) - f(u)).$$

A natural generalization of Laplacian operator is p-Laplacian. For $p > 1$, the *p-Laplacian* of G [3] is the non-linear operator

$$L_p : \mathbb{R}^V \to \mathbb{R}^V, \quad (L_pf)(v) = \sum_{u\sim v} \mathrm{sign}(f(v) - f(u))|f(v) - f(u)|^{p-1},$$

where $\mathrm{sign}(x)$ is the sign-function of a number x, which is 1 if $x > 0$, -1 if $x < 0$ and 0 otherwise. If $f \in \mathbb{R}^V$ and $\mu \in \mathbb{R}$ satisfy $(L_pf)(v) = \mu\,\mathrm{sign}(f(v))|f(v)|^{p-1}$ for every $v \in V$,

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then μ is called an *eigenfunction* (or *eigenvector*) of L_p with the *eigenvalue* μ. The discrete p-Laplacian, which is the analogue of the p-Laplacian on Riemannian manifolds, has been investigated by many researchers. In Ref. [4], Takeuchi investigated the spectrum of the p-Laplacian and the p-harmonic morphism of graphs and proved a Cheeger-type inequality and a Brooks-type inequality for infinite graphs. In Ref. [3], Amghibech presented several sharp upper bounds for the largest p-Laplacian eigenvalues of graphs. In Ref. [5], Bühler and Hein provided a rigorous proof of the approximation of the second eigenvector of p-Laplacian to the Cheeger cut. In Ref. [6], Luo et al. proposed full eigenvector analysis of p-Laplacian and obtained a natural global embedding for multi-class clustering problems. Borba et al. [7] derived many interesting properties for p-Laplacian of a graph.

Note that the signless Laplacian $Q = D + A$ is also a linear operator in $\mathbb{R}^V$ such that $(Qf)(v) = \sum_{u\sim v}(f(u) + f(v))$. Recently, Borba and Schwerdtfeger [8] investigate the so-called signless p-Laplacian. For $p > 1$, the *signless p-Laplacian* is the non-linear operator

$$Q_p : \mathbb{R}^V \to \mathbb{R}^V, \quad (Q_pf)(v) = \sum_{u\sim v} \operatorname{sign}(f(v) + f(u))|f(v) + f(u)|^{p-1}.$$

Similarly, a function f is called an eigenfunction (or eigenvector) of Q_p with eigenvalue λ if $(Q_pf)(v) = \lambda \operatorname{sign}(f(v))|f(v)|^{p-1}$ for every $v \in V$. The energy functional for signless p-Laplacian Q_p is defined to be

$$E_pf = \sum_{uv\in E} |f(u) + f(v)|^p$$

and the l^p norm of a function f is $\|f\|_p = (\sum_v |f(v)|^p)^{1/p}$. The largest eigenvalue of Q_p is

$$\lambda_p = \sup_{f\neq 0} \frac{E_pf}{\|f\|_p^p} = \sup_{f\neq 0} \frac{\sum_{uv\in E} |f(u) + f(v)|^p}{\sum_{v\in V} |f(v)|^p}.$$

In Ref. [8], Borba and Schwerdtfeger gave a Perron–Frobenius type property for λ_p and obtained some basic inequalities for λ_p.

Two distinct edges in graph G are independent if they are not incident with a common vertex in G. A set of pairwise independent edges in G is called a matching in G. The *matching number* $\beta(G)$ (or just β, for short) of G is the cardinality of a maximum matching of G. A *perfect matching* of G is a matching in which every vertex of G is incident to exactly one edge of the matching. It is well known that $\beta(G) \leq \frac{n}{2}$ with equality if and only if G has a perfect matching. Given a vertex subset S of G, the *subgraph induced by S* is denoted by $G[S]$. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two disjoint graphs. The *union* $G_1 \cup G_2$ is defined to be $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. The *join* $G_1 \vee G_2$ of G_1 and G_2 is obtained from $G_1 \cup G_2$ by adding all edges between G_1 and G_2. The *components* of graph G are its maximal connected subgraphs. Components of *odd* (even) order are called odd (even) components. For more notations in graph theory, we refer the reader to Ref. [1].

Feng et al. [9] investigated the adjacency spectral radius of graphs with a given matching number. Yu [10] investigated the signless Laplacian spectral radius of graphs with given matching number. Motivated by the above results, we investigate the signless p-Laplacian spectral radius of graphs with given matching number and generalize Yu's result in Ref. [10]. For convenience, we denote by $\delta(f)$ the root of any continuous monotonic function f.

We shall always assume p and q are conjugate exponents, that is $(p-1)(q-1)=1$. Now we state the main result of this paper.

Theorem 1.1: *Let $\mathcal{G}_{n,\beta}$ be the set of graphs on n vertices with matching number β and let $n_0 = \frac{(2^q-1)^{p-1}(\beta+1)}{2} + \beta$. For any $G \in \mathcal{G}_{n,\beta}$, then the following statements hold.*

(i) *If $n = 2\beta$ or $n = 2\beta+1$, then $\lambda_p(G) \le \lambda_p(K_n)$ with equality if and only if $G = K_n$.*
(ii) *If $2\beta+2 \le n < n_0$, then $\lambda_p(G) \le 2^p\beta$ with equality if and only if $G = K_{2\beta+1} \cup \overline{K_{n-2,\beta-1}}$.*
(iii) *If $n = n_0$, then $\lambda_p(G) \le 2^p\beta$ with equality if and only if $G = K_\beta \vee \overline{K_{n-\beta}}$ or $G = K_{2\beta+1} \cup \overline{K_{n-2,\beta-1}}$.*
(iv) *If $n > n_0$, then $\lambda_p(G) \le \delta(g)$, with equality if and only if $G = K_\beta \vee \overline{K_{n-\beta}}$, where $\delta(g)$ is the unique root of*

$$g(\lambda) = \lambda - 2^{p-1}(\beta-1) - \frac{(n-\beta)\lambda}{(\lambda^{q-1}-\beta^{q-1})^{p-1}} = 0.$$

In the rest of this paper, we will give a proof of Theorem 1.1.

2. The proof of Theorem 1.1

As a start, we need the Perron–Frobenius property for λ_p.

Lemma 2.1 ([8]): *For $p > 1$, let G be a connected graph and f an eigenfunction for λ_p. Then $\lambda_p > 0$ and f is either strictly positive, i.e. $f(v) > 0$ for all $v \in V(G)$, or strictly negative.*

As usual, the strictly positive one is called the *Perron vector* of Q_p.

Borrowing the idea of Lemma 2.7 in Ref. [8], we get the following result from Lemma 2.1.

Lemma 2.2: *For two vertices $u, v \in V(G)$, if $u \not\sim v$, then $\lambda_p(G) < \lambda_p(G+uv)$.*

In fact, Borba and Schwerdtfeger got the following bounds for λ_p in Ref. [8].

Lemma 2.3 ([8]): *For $p > 1$, let $G = (V,E)$ be connected with maximum degree Δ and minimum degree δ. Then*

$$2^{p-1}\delta \le 2^{p-1}\frac{2|E|}{|V|} \le \lambda_p(G) \le 2^{p-1}\Delta,$$

with equality holds if and only if G is regular. In particular, the all one vector is an eigenvector if and only if G is regular.

In what follows, we always assume that $G_0 \in \mathcal{G}_{n,\beta}$ maximizes λ_p and f is the eigenvector corresponding to λ_p. Note that f is the Perron vector only when G_0 is connected. Now we present a property of graphs in $\mathcal{G}_{n,\beta}$.

Lemma 2.4 (The Berge Formula, [1]): *If $G \in \mathcal{G}_{n,\beta}$, then there exists a set S on s vertices in G such that $G - S$ has $r = n + s - 2\beta$ odd components.*

From Lemma 2.4, we get the following result.

Lemma 2.5: *There exist positive odd integers $n_1, \ldots, n_r$ and an integer s satisfying $s = r + 2\beta - n$ and $\sum_{i=1}^{r} n_i = n - s$ such that $G_0 = K_s \vee (\cup_{i=1}^{r} K_{n_i})$.*

Proof: By Lemma 2.4, there exists a subset S on s vertices in G_0 such that $G_0 - S$ has $r = n + s - 2\beta$ odd components. Let $G_1, G_2, \ldots, G_r$ be the odd components in $G_0 - S$ with $|V(G_i)| = n_i \geq 1$ for $1 \leq i \leq r$, and we may assume that $n_1 \leq n_2 \leq \cdots \leq n_r$. It is clear that $n \geq s + r = n + 2s - 2\beta$, and thereby $s \leq \beta$. We divide the remaining proof into two cases to discuss.

Case *1.* $G_0 - S$ contains no even component.

In this case, we have $\cup_{i=1}^{r} V(G_i) = V(G_0) - S$. It suffices to show that $G_0[S]$ is complete, G_i is complete for $1 \leq i \leq r$, and each vertex in S is adjacent to every vertex of G_i for $1 \leq i \leq r$. In fact, if, for example, $G_0[S]$ is not complete, then we could add an edge in $G_0[S]$ to get another graph $G' \in \mathcal{G}_{n,\beta}$, and Lemma 2.2 indicates $\lambda_p(G') > \lambda_p(G_0)$, a contradiction.

Case 2. $G_0 - S$ contains an even component.

Let C be the union of these even components. By adding edges to make $G[G_r \cup C]$ to be complete, it results in a graph $G' \in \mathcal{G}_{n,\beta}$, and Lemma 2.2 implies $\lambda_p(G) < \lambda_p(\widetilde{G})$, a contradiction. ■

Lemma 2.5 gives a rough structure of G_0, and now we will say more for the structure of G_0.

Lemma 2.6: *There exist integers s and r satisfying $r = n + s - 2\beta$ and $n_r = 2\beta - 2s + 1$ such that $G_0 = K_s \vee (K_{n_r} \cup \overline{K_{r-1}})$.*

Proof: According to Lemma 2.5, we only need to prove that $n_1 = n_2 = \cdots = n_{r-1} = 1$ and $n_r = 2\beta - 2s + 1$. Recall that f is the eigenvector of $Q_p(G_0)$ corresponding to λ_p. We divide the remaining proof into two cases to discuss.

Case 1. $s = 0$.

In this case, $G_0 = \cup_{i=1}^{r} K_{n_i}$. We first claim that $f(u)f(v) \geq 0$ for any two vertices u, v in the same component. Otherwise, there exists a vertex set partition V^+, V^-, V^0 of some $V(K_{n_i})$ such that $V^+ = \{v \in V(K_{n_i}) \mid f(v) > 0\}$, $V^- = \{v \in V(K_{n_i}) \mid f(v) < 0\}$ and $V^0 = \{v \in V(K_{n_i}) \mid f(v) = 0\}$. Let f' be the function such that $f'(v) = |f(v)|$ for $v \in V(G_0)$. It is clear that $\|f'\|_p = \|f\|_p$. Since $|f(u) + f(v)|^p < |f'(u) + f'(v)|^p$ for $u \in V^+$ and $v \in V^-$, we have $E_p f < E_p f'$ and hence $\lambda_p = \frac{E_p f}{\|f\|_p^p} < \frac{E_p f'}{\|f'\|_p^p}$, a contradiction.

Next, we claim $f(u) = f(v)$ for two vertices u, v in the same component $V(K_{n_i})$. Otherwise, assume that $f(u) \neq f(v)$, and we define f' by $f'(u) = f'(v) = (\frac{f(u)^p + f(v)^p}{2})^{1/p}$ and $f'(w) = f(w)$ for any $w \neq u, v$. Clearly, $\|f'\|_p = \|f\|_p$. By using Jensen's inequality to convex function t^p, we have

$$(f(u) + f(v))^p < 2^{p-1}(f(u)^p + f(v)^p) = (f'(u) + f'(v))^p,$$

and, for any $w \in V(K_{n_i}) \setminus \{u, v\}$,

$$(f(u) + f(w))^p + (f(v) + f(w))^p = \left\| \begin{matrix} f(u) + f(w) \\ f(v) + f(w) \end{matrix} \right\|_p^p$$

$$
\begin{aligned}
&= \left\| \begin{pmatrix} f(u) \\ f(v) \end{pmatrix} + \begin{pmatrix} f(w) \\ f(w) \end{pmatrix} \right\|_p^p \le \left(\left\| \begin{matrix} f(u) \\ f(v) \end{matrix} \right\|_p + \left\| \begin{matrix} f(w) \\ f(w) \end{matrix} \right\|_p \right)^p \\
&= \left((f(u)^p + f(v)^p)^{\frac{1}{p}} + 2^{\frac{1}{p}} f(w) \right)^p = 2 \left[\left(\frac{f(u)^p + f(v)^p}{2} \right)^{\frac{1}{p}} + f(w) \right]^p \\
&= (f'(u) + f'(w))^p + (f'(v) + f'(w))^p.
\end{aligned}
$$

Thus, we have

$$E_p f = \sum_{ij \in E(G_0)} |(f(i) + f(j)|^p < \sum_{ij \in E(G_0)} |f'(i) + f'(j)|^p = E_p f',$$

which leads to $\lambda_p = \frac{E_p f}{\|f\|_p^p} < \frac{E_p f'}{\|f'\|_p^p}$, a contradiction.

Assume that $f(v_i) = x_i$ for any $v_i \in K_{n_i}$ and $1 \le i \le r$. We have

$$\lambda_p = \frac{\sum_i \frac{n_i(n_i-1)}{2} |2x_i|^p}{\sum_i n_i |x_i|^p} = 2^{p-1} \left(\frac{\sum_i n_i^2 |x_i|^p}{\sum_i n_i |x_i|^p} - 1 \right) \le 2^{p-1}(n_r - 1),$$

where the equality holds if $x_1 = x_2 = \cdots = x_{r-1} = 0$. This leads to $\lambda_p(G_0) = 2^{p-1}(n_r - 1)$. This means that if we increase n_r by 2 and decrease n_i by 2 for $1 \le i \le r-1$, the signless p-Laplacian spectral radius will increase; moreover, the resulting graph still has matching number β. Therefore, $n_1 = n_2 = \cdots = n_{r-1} = 1$ and $n_r = 2\beta + 1 = 2\beta - 2s + 1$.

Case 2. $s \neq 0$.

In this case, G_0 is connected and f is the Perron vector. By the same arguments as those in Case 1, we may assume $f(v_i) = x_i$ for any $v_i \in V(K_{n_i})$ and $1 \le i \le r$, and $f(v) = y$ for any $v \in S$. From the eigenequations we have

$$
\begin{cases}
\lambda_p x_i^{p-1} = (n_i - 1)(x_i + x_i)^{p-1} + s(x_i + y)^{p-1}, \\
\lambda_p y^{p-1} = (s-1)(y + y)^{p-1} + \sum_{i=1}^r n_i (x_i + y)^{p-1}.
\end{cases}
$$

From the first equation, we have $\lambda_p = 2^{p-1}(n_i - 1) + s(1 + \frac{y}{x_i})^{p-1}$, and thus $x_i \le x_j$ whenever $n_i \le n_j$. Hence $x_1 \le x_2 \le \cdots \le x_r$ since $n_1 \le n_2 \le \cdots \le n_r$. If $n_{r-1} \ge 3$, then denote by

$$G' = K_s \vee (K_{n_r+2} \cup K_{n_{r-1}-2} \cup_{i=1}^{r-2} K_{n_i}).$$

It is clear that $G' \in \mathcal{G}_{n,\beta}$ and G' could be viewed as the graph obtained from G_0 by 'removing' two vertices of $K_{n_{r-1}}$ to K_{n_r}. Without loss of generality, assuming that $\|f\|_p = 1$, and now we have

$$
\begin{aligned}
\lambda_p(G') - \lambda_p(G_0) &\ge E_p(G')f - E_p(G_0)f \\
&= 2n_r(x_{r-1} + x_r)^p - 2(n_{r-1} - 2)(x_{r-1} + x_{r-1})^p > 0,
\end{aligned}
$$

a contradiction. Hence $n_{r-1} \le 2$. Similarly, we get $n_1 = n_2 = \cdots = n_{r-1} = 1$ and $n_r = 2\beta - 2s + 1$.

This completes the proof. ■

Since $0 \leq s \leq \beta$, Lemma 2.6 indicates that G_0 may have $\beta + 1$ different choices according to the value of s, and we will determine the exact G_0 according to the value of s. As discussed above, if $s = 0$, then G_0 is disconnected; if $1 \leq s \leq \beta$, then G_0 is connected. Denote by

$$H_1 = K_{2\beta+1} \cup \overline{K_{n-2\beta-1}} \quad \text{and } H_2 = K_\beta \vee \overline{K_{n-\beta}}.$$

Then $\lambda_p(H_1) = 2^p\beta$. Now, we investigate $\lambda_p(H_2)$.

Lemma 2.7: *The largest signless p-Laplacian eigenvalue $\lambda_p(H_2)$ is the unique root of*

$$g(\lambda) = \lambda - 2^{p-1}(\beta - 1) - \frac{(n-\beta)\lambda}{(\lambda^{q-1} - \beta^{q-1})^{p-1}} = 0.$$

In particular, when $n = n_0 = \frac{(2^q-1)^{p-1}(\beta+1)}{2} + \beta$, we have $\lambda_p(H_2) = 2^p\beta$. Moreover, if $n < n_0$, then $\lambda_p(H_2) < 2^p\beta$; if $n > n_0$, then $\lambda_p(H_2) > 2^p\beta$.

Proof: Let f be the Perron vector of $\lambda_p(H_2)$. By the same arguments as in Lemma 2.6, all vertices in K_β or $\overline{K_{n-\beta}}$ have the same eigencomponents, and we assume $f(u) = x_1$ for any $u \in K_\beta$ and $f(v) = x_2$ for any $v \in \overline{K_{n-\beta}}$. Then, from the eigenequations, we have

$$\begin{cases} \lambda_p x_1^{p-1} = (\beta-1)(x_1+x_1)^{p-1} + (n-\beta)(x_1+x_2)^{p-1}, \\ \lambda_p x_2^{p-1} = \beta(x_1+x_2)^{p-1}. \end{cases}$$

It leads to that

$$\lambda_p - 2^{p-1}(\beta-1) - \frac{(n-\beta)\lambda_p}{(\lambda_p^{q-1} - \beta^{q-1})^{p-1}} = 0.$$

Note that the function $g(\lambda) = \lambda - 2^{p-1}(\beta-1) - \frac{(n-\beta)\lambda}{(\lambda^{q-1}-\beta^{q-1})^{p-1}}$ is continuous, its derivative is $g'(\lambda) = 1 + (n-\beta)\frac{\beta^{q-1}}{(\lambda^{q-1}-\beta^{q-1})^p} > 0$, $\lim_{\lambda\to\beta^+} g(\lambda) = -\infty$ and $\lim_{\lambda\to+\infty} g(\lambda) = +\infty$. We conclude that $\lambda_p(H_2)$ is the unique solution of $g(\lambda)$. Taking $n = n_0 = \frac{(2^q-1)^{p-1}(\beta+1)}{2} + \beta$, we have $\lambda_p(H_2) = 2^p\beta$, and thus $\lambda_p(H_2) < 2^p\beta$ if $n < n_0$ and $\lambda_p(H_2) > 2^p\beta$ if $n > n_0$. ■

Now we consider the relation between β, n and n_0.

Lemma 2.8: *If $2\beta + 2 \leq n < n_0$, then $\lambda_p(G_0) \leq 2^p\beta$ with equality if and only if $G_0 = H_1$.*

Proof: Since $\lambda_p(H_1) = 2^p\beta$, it suffices to prove that $\lambda_p(G_0) < 2^p\beta$ when $G_0 \neq H_1$. From Lemma 2.6, we have $G_0 = K_s \vee (K_{2\beta-2s+1} \cup \overline{K_{n+s-2\beta-1}})$. Since $G_0 \neq H_1$, we have $s \geq 1$.

Assume that $f(u) = x_1$ for any $u \in K_s$, $f(v) = x_2$ for any $v \in K_{2\beta-2s+1}$ and $f(w) = x_3$ for any $w \in \overline{K_{n+s-2\beta-1}}$. From the eigenequations we have

$$\begin{cases} \lambda_p x_1^{p-1} &= (s-1)(x_1+x_1)^{p-1} + (2\beta-2s+1)(x_1+x_2)^{p-1} \\ & \quad +(n+s-2\beta-1)(x_1+x_3)^{p-1}, \\ \lambda_p x_2^{p-1} &= (2\beta-2s)(x_2+x_2)^{p-1} + s(x_1+x_2)^{p-1}, \\ \lambda_p x_3^{p-1} &= s(x_1+x_3)^{p-1}. \end{cases}$$

From the second and the third eigenequation, we have $\frac{x_1}{x_2} = (\frac{\lambda_p - 2^p(\beta-s)}{s})^{q-1} - 1$ and $\frac{x_1}{x_3} = (\frac{\lambda_p}{s})^{q-1} - 1$. Combining with the first eigenequation we have that

$$\lambda_p = 2^{p-1}(s-1) + \frac{(n+s-2\beta-1)\lambda_p}{(\lambda_p^{q-1} - s^{q-1})^{p-1}} + \frac{(2\beta-2s+1)[\lambda_p - 2^p(\beta-s)]}{[(\lambda_p - 2^p(\beta-s))^{q-1} - s^{q-1}]^{p-1}}.$$

Thus $\lambda_p(G_0)$ is the solution of

$$h(\lambda) = \lambda - 2^{p-1}(s-1) - \frac{(n+s-2\beta-1)\lambda}{(\lambda^{q-1} - s^{q-1})^{p-1}} - \frac{(2\beta-2s+1)[\lambda - 2^p(\beta-s)]}{[(\lambda - 2^p(\beta-s))^{q-1} - s^{q-1}]^{p-1}} = 0.$$

One can verify that $h(\lambda)$ is a continuous and monotonically increasing function on $(2^p(\beta-s)+s, +\infty)$, and $h(\lambda) = 0$ has exactly one solution $\delta(h)$. Hence $\lambda_p(G_0) = \delta(h)$.

Next, we claim that if $n < n_0$, then $h(2^p\beta) > 0$. In fact, bearing in mind that $(p-1)(q-1) = 1$, we have

$$\begin{aligned} h(2^p\beta) &= 2^p\beta - 2^{p-1}(s-1) - \frac{2^p\beta(n+s-2\beta-1)}{[2^q\beta^{q-1} - s^{q-1}]^{p-1}} - \frac{2^p s(2\beta-2s+1)}{[(2^p s)^{q-1} - s^{q-1}]^{p-1}} \\ &= 2^p\left(\beta - \frac{s-1}{2} - \frac{\beta(n+s-2\beta-1)}{[2^q\beta^{q-1} - s^{q-1}]^{p-1}} - \frac{2\beta-2s+1}{(2^q-1)^{p-1}}\right) \\ &= 2^p\left(\beta - \frac{s-1}{2} - \frac{n+s-2\beta-1}{(2^q - (\frac{s}{\beta})^{q-1})^{p-1}} - \frac{2\beta-2s+1}{(2^q-1)^{q-1}}\right) \\ &\geq 2^p\left(\beta - \frac{s-1}{2} - \frac{n+s-2\beta-1}{(2^q-1)^{p-1}} - \frac{2\beta-2s+1}{(2^q-1)^{q-1}}\right) \\ &= 2^p\left(\beta - \frac{s-1}{2} - \frac{n-s}{(2^q-1)^{p-1}}\right) \\ &> 2^p\left(\beta - \frac{s-1}{2} - \frac{\frac{(2^q-1)^{p-1}}{2}(\beta+1) + \beta - s}{(2^q-1)^{p-1}}\right) \\ &= 2^p\left(\beta - \frac{s-1}{2} - \frac{\beta+1}{2} - \frac{\beta-s}{(2^q-1)^{p-1}}\right) \\ &= 2^p\left(\frac{1}{2} - \frac{1}{(2^q-1)^{p-1}}\right)(\beta-s) \\ &\geq 0, \end{aligned}$$

where the three inequalities respectively follow from $\frac{s}{\beta} \leq 1$, $n < n_0 = \frac{(2^q-1)^{p-1}(\beta+1)}{2} + \beta$ and $(2^q-1)^{p-1} = (2^{q-1}+2^{q-1}-1)^{p-1} > (2^{q-1})^{p-1} = 2$. Hence $\lambda_p(G_0) = \delta(h) < 2^p\beta$. ∎

Lemma 2.9: *If $n > n_0$, then $\lambda_p(G_0) \leq \delta(g)$, where $\delta(g)$ is the only solution to $g(\lambda) = \lambda - 2^{p-1}(\beta-1) - \frac{(n-\beta)\lambda}{(\lambda^{q-1}-\beta^{q-1})^{p-1}} = 0$, with equality if and only if $G_0 = H_2$.*

Proof: From Lemma 2.7, $\lambda_p(H_2) = \delta(g) > 2^p\beta$. Since $\lambda_p(H_1) = 2^p\beta < \lambda_p(H_2)$, we may assume $1 \leq s \leq \beta$, and thereby $\lambda_p(G_0) = \delta(h)$. It suffices to show that $\delta(h) \leq \delta(g)$ and the equality holds only when $s = \beta$.

We now consider $h(\lambda)$ in the interval $(2^p\beta, +\infty)$. It is obvious that $\frac{s}{\lambda} \leq \frac{s}{\lambda - 2^p(\beta-s)} \leq \frac{\beta}{\lambda} < 1$ for any $\lambda \in (2^p\beta, +\infty)$ with the first two equalities hold only when $s = \beta$. We claim $h(\lambda) \geq g(\lambda)$ for any $\lambda \in (2^p\beta, +\infty)$, with equality holds only when $s = \beta$. In fact,

$$\begin{aligned}
h(\lambda) &= \lambda - 2^{p-1}(s-1) - \frac{n+s-2\beta-1}{(1-(\frac{s}{\lambda})^{q-1})^{p-1}} - \frac{2\beta-2s+1}{(1-(\frac{s}{\lambda-2^p(\beta-s)})^{q-1})^{p-1}} \\
&\geq \lambda - 2^{p-1}(s-1) - \frac{n+s-2\beta-1}{(1-(\frac{\beta}{\lambda})^{q-1})^{p-1}} - \frac{2\beta-2s+1}{(1-(\frac{\beta}{\lambda})^{q-1})^{p-1}} \\
&= \lambda - 2^{p-1}(s-1) - \frac{n-s}{(1-(\frac{\beta}{\lambda})^{q-1})^{p-1}} \\
&= g(\lambda) + 2^{p-1}(\beta-s) - \frac{\beta-s}{(1-(\frac{\beta}{\lambda})^{q-1})^{p-1}} \\
&= g(\lambda) + (\beta-s)\left(2^{p-1} - \left(\frac{1}{1-(\frac{\beta}{\lambda})^{q-1}}\right)^{p-1}\right) \\
&\geq g(\lambda),
\end{aligned}$$

where the last inequality follows from $(\frac{\beta}{\lambda})^{q-1} < (2^{-p})^{q-1} = 2^{-q} < \frac{1}{2}$. Hence $\delta(h) \leq \delta(g)$ with equality if and only if $s = \beta$. ∎

Now we are ready to prove Theorem 1.1.

Proof: It is easy to see that $G_0 = K_n$ when $n = 2\beta$ or $n = 2\beta+1$. Now we assume that $n \geq 2\beta+2$. If $2\beta+2 \leq n < n_0$, then Lemma 2.8 implies that $G_0 = H_1 = K_{2\beta+1} \cup \overline{K_{n-2\beta-1}}$. If $n > n_0$, then Lemma 2.9 implies that $G_0 = H_2 = K_\beta \vee \overline{K_{n-\beta}}$. If $n = n_0$, then $\lambda_p(H_1) = \lambda_p(H_2) = 2^p\beta$. Moreover, when $1 \leq s \leq \beta-1$, by the same discussion as in Lemma 2.8, we have that $h(2^p\beta) > 0$. It implies that $\lambda_p(G_0) = \delta(h) < 2^p\beta$ when $1 \leq s \leq \beta-1$. Hence $\lambda_p(G_0) \leq 2^p\beta$ with equality if and only if $G_0 = H_1$ or $G_0 = H_2$. ∎

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