Discrete Mathematics 345 (2022) 112939

Contents lists available at ScienceDirect

Discrete Mathematics

www.elsevier.com/locate/disc

Mixed graphs with smallest eigenvalue greater than $-\frac{\sqrt{5}+1}{2}$

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ARTICLE INFO

Article history:
Received 28 June 2021
Received in revised form 29 March 2022
Accepted 12 April 2022
Available online xxxx

Keywords:
Mixed graph
Hermitian matrix
Smallest eigenvalue

ABSTRACT

The classical problem of characterizing the graphs whose eigenvalues lie in a given interval may date back to the work of J.H. Smith in 1970. Especially, the research on graphs with smallest eigenvalues not less than -2 has attracted widespread attention. Mixed graphs are natural generalizations of undirected graphs. In this paper, we completely characterize the mixed graphs with smallest Hermitian eigenvalue greater than $-\frac{\sqrt{5}+1}{2}$. In fact, we found three infinite classes of mixed graphs and 30 scattered mixed graphs.

1. Introduction

It is a classical problem in Spectral Graph Theory to characterize the graphs whose eigenvalues lie in a given interval. The research of such problems may date back to the work of Smith in 1970 [18]. This work stimulated the interest of researchers. There are a lot of results in the literature concerning the topic. In 1972, Hoffman [11] obtained all limit points of the spectral radius of nonnegative symmetric matrices smaller than $\frac{\sqrt{5}+1}{2}$. In 1982, Cvetković et al. [5] characterized the graphs whose spectral radius does not exceed $\sqrt{2+\sqrt{5}}$, in 1989, Brouwer and Neumaier [4] determined the graphs with spectral radius between 2 and $\sqrt{2+\sqrt{5}}$ and later, Woo and Neumaier [27] described the structure of graphs whose spectral radii are bounded above by $3\sqrt{2}/2$. For the (signless) Laplacian matrices, Wang et al. [21,22] characterized the graphs whose spectral radii do not exceed 4.5. With respect to the smallest eigenvalues, Hoffman [12] investigated the graphs whose smallest eigenvalue exceeds $-1-\sqrt{2}$, and this work was continued by Taniguchi et al. [19,20,14]. Furthermore, Munemasa et al. [16] showed that all fat Hoffman graphs with smallest eigenvalue at least $-\frac{\sqrt{5}+1}{2}$ (which is just $-1-\tau$ where τ is the golden ratio) can be described by a finite set of fat $(-1-\tau)$-irreducible Hoffman graphs. Especially, the graphs with smallest eigenvalue -2 attracted a lot of attention, and we refer the reader to the survey [7] and the book [6]. Recently, Abdollahi et al. [1] classified all distance-regular Cayley graphs with least eigenvalue -2 and diameter at most three, and Koolen et al. [13] proved that a connected graph with smallest eigenvalue at least -3 and large enough minimal degree is 2-integrable. For the anti-adjacency matrices, Wang et al. [23,24] determined the graphs whose smallest eigenvalues are at least $-2\sqrt{2}$. In this paper, we consider the smallest Hermitian eigenvalue of a mixed graph.

A mixed graph is defined to be an ordered triple (V, E, A), where V is the vertex set, E is the undirected edge set and A is the directed edge set. Note that, if both uv and vu are directed edges, then we regard $\{u, v\}$ as an undirected edge. Thus, if $(u, v) \in A$ then $(v, u) \notin A$. Clearly, if $A = \emptyset$ then the mixed graph turns to be a graph and if $E = \emptyset$ then the mixed graph

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https://doi.org/10.1016/j.disc.2022.112939
0012-365X/

turns to be an oriented graph. For convenience, we write $u \leftrightarrow v$ if $\{u, v\} \in E$ and $u \to v$ is $(u, v) \in A$. Let $M = (V, E, A)$ be a mixed graph with $V = \{v_1, v_2, \dots, v_n\}$. The *underlying graph* $\Gamma(M)$ is a graph with vertex set V and two vertices $u \sim v$ if $u \leftrightarrow v$ or $u \to v$ or $v \to u$. For $U \subseteq V$ and $W \subseteq V \setminus U$, denote by $N_W(U) = \{w \mid w \in W, u \sim w \text{ in } \Gamma(M) \text{ for some } u \in U\}$. Especially, if $U = \{u\}$ then $N_W(u)$ is the set of neighbors of u in W. Moreover, denote by $N_W^+(u) = \{w \mid u \to w\}$, $N_W^-(u) = \{w \mid u \leftarrow w\}$ and $N_W^o(u) = \{w \mid u \leftrightarrow w\}$. It is clear that $N_W(u) = N_W^+(u) \cup N_W^-(u) \cup N_W^o(u)$. As usual, we always write P_n, C_n, $K_{n_1,n_2,\dots,n_k}$ to denote the path, the cycle, and the complete multipartite graph of the corresponding orders. For two graphs G and H, the *union* $G \cup H$ is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. The *join* $G \nabla H$ is the graph obtained from $G \cup H$ by adding all edges between G and H. The *distance* $d_G(u, v)$ of two vertices $u, v \in V(G)$ is the length of a shortest path from u to v in G. The *diameter* of G is the largest distance in G, denoted by $d(G)$. All other notations not mentioned here are standard and can be found in [8].

We always write M_G for M when the underlying graph $\Gamma(M) = G$. Moreover, for a graph G, denote by $\mathcal{M}_G$ the set of mixed graphs with underlying graph G. If $M_G = G$ then we write G for M_G. The mixed graph M_G is *connected* if G is connected and we always consider connected mixed graphs in this paper. The *diameter* of M_G is defined to be the diameter of G, denoted by $d(M_G)$. For a subset $U \subseteq V$, the mixed subgraph induced by U is the mixed graph $M_G[U] = (U, E', A')$ with $E' = \{\{u, v\} \mid u, v \in U, \{u, v\} \in E\}$ and $A' = \{(u, v) \mid u, v \in U, (u, v) \in A\}$. As usual, for a vertex v, the (mixed) graph $G - v$ (resp. $M_G - v$) is the induced (resp. mixed) subgraph obtained from G (resp. M_G) by deleting the vertex v and associated edges. The *Hermitian matrix* of M_G is defined to be a square matrix $H(M_G) = [h_{st}]_{n\times n}$ with

$$h_{st} = \begin{cases} 1, & v_s \leftrightarrow v_t, \\ \mathbf{i}, & v_s \to v_t, \\ -\mathbf{i}, & v_t \to v_s, \\ 0, & \text{otherwise}, \end{cases}$$

where $\mathbf{i} = \sqrt{-1}$. This matrix was proposed by Liu and Li [15] and Guo and Mohar [9] independently. Since $H(M_G)$ is a Hermitian matrix, all eigenvalues of $H(M_G)$ are real and listed as $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n = \lambda_{\min}$. The collection of such eigenvalues is the spectrum of $H(M_G)$. The *Hermitian spectrum* of the mixed graph M_G is just the spectrum of $H(M_G)$, denoted by $\mathrm{Sp}(M_G)$. Two mixed graphs $M_G, M_G' \in \mathcal{M}_G$ are *switching equivalent* if there exists a diagonal matrix D whose entries belong to $\{\pm 1, \pm \mathbf{i}\}$ such that $H(M_G') = DH(M_G)D^*$. It is clear that switching equivalence is an equivalence relation. Thus, denote by $[M_G]$ the equivalence class containing M_G with respect to switching equivalence. Obviously, all graphs in $[M_G]$ share the same spectrum. Recently, Wissing and Dam [26] determined all mixed graphs with exactly one negative eigenvalue. Guo and Mohar [10] determined all mixed graphs with $\lambda_1 < 2$ and Yuan et al. [28] characterized all mixed graphs with $\lambda_1 \le 2$ when G contains no cycles of length 4.

In this paper, we completely determine the connected mixed graphs with smallest Hermitian eigenvalue greater than $-\frac{\sqrt{5}+1}{2}$. In fact, we found three infinite classes of mixed graphs and 30 scattered mixed graphs (see Theorem 5).

2. Preliminaries

We first present the famous interlacing theorem with respect to Hermitian matrices.

Lemma 1 *([3]). Let the matrix S of size $n \times m$ be such that $S^*S = I_m$ and let H be a Hermitian matrix of size n with eigenvalues $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_{\min}$. Set $B = S^*HS$ and let $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_m$ be the eigenvalues of B. Then the eigenvalues μ_i interlace the eigenvalues λ_j, that is, $\lambda_i \ge \mu_i \ge \lambda_{n-m+i}$ for $i = 1, 2, \dots, m$.*

The following result is immediate from Lemma 1.

Corollary 1. *Let M_G be a mixed graph with underlying graph G. If M_H is a mixed induced subgraph of M_G, then the eigenvalues of M_H interlace those of M_G.*

Next, we introduce another powerful tool in Spectral graph theory: the equitable partition. Let M_G be a mixed graph on n vertices with underlying graph G. Let $\pi\colon V(G) = V_1 \cup V_2 \cup \cdots \cup V_s$ be a partition of $V(G)$ with $|V_i| = n_i$ and $n = n_1 + n_2 + \cdots + n_s$. For $1 \le i, j \le s$, denote by H_{ij} the submatrix of $H(M_G)$ whose rows correspond to V_i and columns correspond to V_j. Therefore, the Hermitian matrix $H(M_G)$ can be written as $H(M_G) = [H_{ij}]$. Denote by $b_{ij} = \mathbf{1}^T H_{ij} \mathbf{1}/n_i$ the average row-sums of H_{ij}, where $\mathbf{1}$ denotes the all-one vector. The matrix $H_\pi = (b_{ij})_{s\times s}$ is called the *quotient matrix* of $H(M_G)$. If, for any i, j, the row-sum of H_{ij} corresponding to any vertex $v \in V_i$ equals to b_{ij}, then π is called an *equitable partition* of M_G. Let δ_{V_i} be a vector indexed by $V(G)$ such that $\delta_{V_i}(v) = 1$ if $v \in V_i$ and 0 otherwise. The matrix $P = [\delta_{V_1} \delta_{V_2} \cdots \delta_{V_s}]$ is called the *characteristic matrix* of π. If π is an equitable partition, then $H(M_G)P = PH_\pi$. It leads to the following famous result.

Lemma 2 *([8, Theorem 9.3.3, page 197]). Let M_G be a mixed graph and π an equitable partition of M_G with quotient matrix H_π and characteristic matrix P. Then the eigenvalues of H_π are also eigenvalues of $H(M_G)$. Furthermore, $H(M_G)$ has the following two kinds of eigenvectors:*

(i) *the eigenvectors in the column space of* P*, and the corresponding eigenvalues coincide with the eigenvalues of* H_π*;*
(ii) *the eigenvectors orthogonal to the columns of* P*, i.e., those eigenvectors sum to zero on each cell of* π*.*

Let $\mathcal{H}$ be a set of graphs. A graph G is called $\mathcal{H}$-free if no induced subgraphs of G are in $\mathcal{H}$. If $\mathcal{H} = \{H\}$ then an $\mathcal{H}$-free graph G is also called an H-free graph. Recall that a P_4-free graph is called a cograph. The following result reveals the structure of cographs.

Lemma 3 *([17]). If* G *is a connected* P_4*-free graph, then* G *is the join of two graphs, that is,* $G = G_1 \nabla G_2$ *for some graphs* G_1 *and* G_2 *with* $|V(G_1)|, |V(G_2)| \geq 1$*.*

We determine two types of $\mathcal{H}$-free graphs when $\mathcal{H}$ contains some simple graphs.

Lemma 4. *If* G *is a* $\{P_3, 3K_1, K_2 \cup K_1\}$*-free graph then* $G \in \{2K_1, K_n \mid n \geq 1\}$*; if* G *is a* $\{P_3, 3K_1, K_3\}$*-free graph then* $G \in \{K_1, K_2, 2K_1, 2K_2, K_1 \cup K_2\}$*.*

Proof. It is clear that, if a graph G is $P_3, 3K_1$-free, then it is the union of at most two complete graphs. Thus, we have $G \in \{2K_1, K_n \mid n \geq 1\}$ if G is additional $K_2 \cup K_1$-free, and $G \in \{K_1, K_2, 2K_1, 2K_2, K_2 \cup K_1, K_2 \cup K_2\}$ if G is additional K_3-free. □

Guo and Mohar introduced the so-called four-way switching to generate switching equivalent graphs [9]. A *four-way switching* is the operation of changing a mixed graph M_G into the mixed graph M'_G by choosing an appropriate diagonal matrix S with $S_{jj} \in \{\pm 1, \pm \mathbf{i}\}$ and setting $H(M'_G) = S^{-1} H(M_G) S$. Let G be a graph and X an edge cut such that $G - X = G_1 \cup G_2$ and $V_1 = V(G_1)$ and $V_2 = V(G_2)$. For a mixed graph $M_G = (V, E, A)$, define $X^+ = \{(v_1, v_2) \mid \{v_1, v_2\} \in X, v_1 \in V_1, v_2 \in V_2\}$ and $X^- = \{(v_2, v_1) \mid \{v_1, v_2\} \in X, v_1 \in V_1, v_2 \in V_2\}$. The cut X is called a *coincident cut* of the mixed graph M_G if $X^+ \subseteq A$ or $X^- \subseteq A$ or $X \subseteq E$. If X is a coincident cut of M_G, then the X*-switching* of M_G is the mixed graph $M_G[X] = (V, E', A')$ with $E' = E \cup X$ and $A' = A \setminus (X^+ \cup X^-)$. Note that $M_G[X] = M_G$ if $X \subseteq E$. From four-way switching, the following results are obtained.

Lemma 5 *([9]). Let* M_G *be a mixed graph. If* X *is a coincident cut of* M_G*, then* M_G *and* $M_G[X]$ *are switching equivalent and thus* $\mathrm{Sp}(M_G) = \mathrm{Sp}(M_G[X])$*.*

If G is a forest, then each edge is a cut. Moreover, each edge is a coincident cut of any mixed graph M_G. Thus, Lemma 5 implies the following result.

Corollary 2 *([9]). If* G *is a forest, then* $\mathrm{Sp}(M_G) = \mathrm{Sp}(G)$ *for any mixed graph* $M_G \in \mathcal{M}_G$*.*

Note that mixed graphs could be viewed as the so-called gain graphs. Let $\mathbb{T}_4 = \{\pm 1, \pm \mathbf{i}\}$ be the group of the fourth roots of unity. For an undirected graph G with vertex set V and edge set E, the $\mathbb{T}_4$-gain graph is a triple $\Phi = (G, \mathbb{T}_4, \varphi)$ consisting the underlying graph G, the gain group $\mathbb{T}_4$ and a map $\varphi\colon \vec{E} \to \mathbb{T}_4$ such that $\varphi((u, v)) = \varphi((v, u))^{-1}$ called the gain function. The mixed graphs are just the $\mathbb{T}_4$-gain graphs with $\varphi(\vec{E}) \in \{1, \pm i\}$. For a mixed graph M_G with Hermitian matrix $H = [h_{uv}]_{n \times n}$, let $C = v_1 v_2 \cdots v_s$ be a cycle in G, denote by $h_{M_G}(C) = h_{v_1 v_2} h_{v_2 v_3} \cdots h_{v_s v_1}$. Moreover, if $h_{M_G}(C) = 1$ then we say C is positive. In [2], the authors investigated $\mathbb{T}_4$-gain graphs, and from Propositions 1 and 2 in [2], we get the following useful result, which is also obtained by Wang and Yuan [25].

Lemma 6 *([2,25]). Let* M_G *and* M'_G *be two mixed graphs sharing the same underlying graph* G*. If every induced cycle* C *in* G *is positive, then* $M_G \in [G]$*.*

3. Mixed graphs with $\lambda_{\min} > -\frac{\sqrt{5}+1}{2}$

In this part, we first investigate the mixed triangles in mixed graphs whose underlying graph is a complete graph. Next, we get all mixed graphs with smallest eigenvalue not less than $-\sqrt{2}$. At last, we completely determine the mixed graphs with smallest eigenvalue greater than $-\frac{\sqrt{5}+1}{2} \approx -1.618$.

It is easy to verify that there are seven types of mixed triangles and fourteen types of mixed quadrangles, and we present them in Fig. 1 together with their smallest eigenvalues. The following results are immediate from Lemma 1 and Fig. 1.

Lemma 7. *Let* M_G *be a mixed graph with smallest eigenvalue* $\lambda_{\min}$*. If* $\lambda_{\min} > -\sqrt{3}$*, then any mixed triangle in* M_G *belongs to* $\{K_3, K_3^{2,2}, K_3^{2,3}\}$*.*

Lemma 8. *Let* M_G *be a mixed graph with smallest eigenvalue* $\lambda_{\min}$*. If* $\lambda_{\min} \geq -1.84$*, then any induced mixed quadrangle in* M_G *belongs to* $\{C_4^1, C_4^2, C_4^3\}$*.*

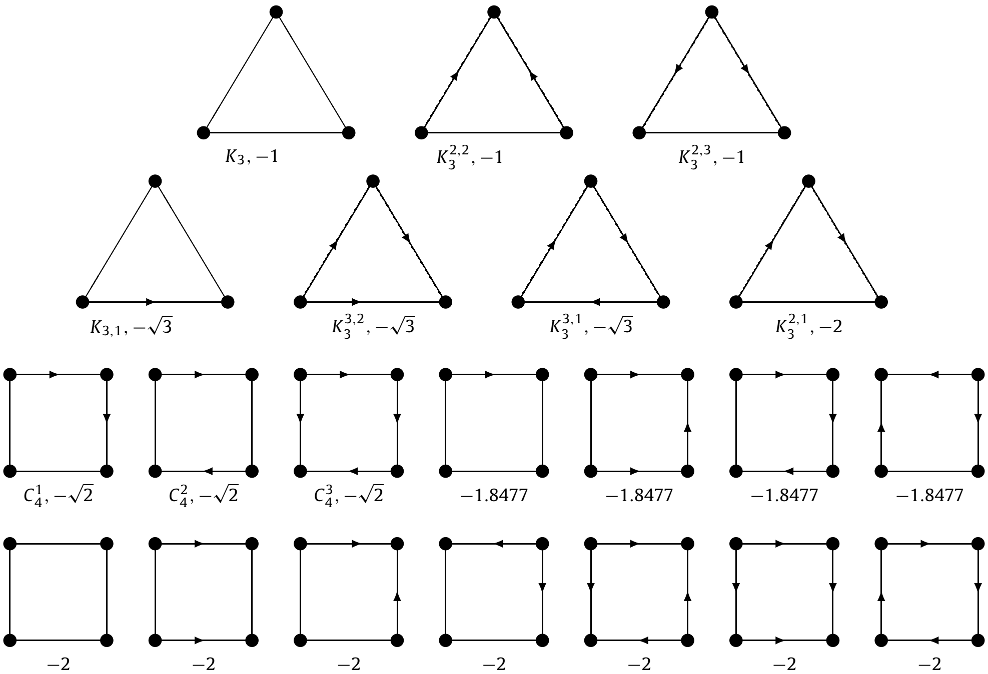


Fig. 1. The mixed triangles and quadrangles together with their smallest eigenvalues, where $K_3, K_3^{2,2}, K_3^{3,2}$ are switching equivalent and $K_{3,1}, K_3^{3,2}$ are switching equivalent. The first to the sixth graphs on the last row are switching equivalent.

In what follows, we always denote $\mathcal{C}_3 = \{K_3, K_3^{2,2}, K_3^{2,3}\}$ and $\mathcal{C}_4 = \{C_4^1, C_4^2, C_4^3\}$. The mixed triangles K_3, $K_3^{2,2}$ and $K_3^{2,3}$ play an important role in determining the orientations of a mixed graph, especially when all induced cycles (if exist) of the underlying graph are triangles. Recall that a *chordal graph* is one in which all cycles of four or more vertices have a chord, which is an edge that is not part of the cycle but connects two vertices of the cycle. The following result characterizes a class of mixed chordal graphs switching equivalent to their underlying graphs.

Theorem 1. *Let G be a chordal graph. If M_G is a mixed graph in which each mixed triangle belongs to $\mathcal{C}_3$, then $M_G \in [G]$, i.e., M_G is switching equivalent to G.*

Proof. Without loss of generality, assume that G is connected. According to Lemma 5, it suffices to show that M_G has a coincident cut X such that $V(G-X) = U \cup W$ satisfying that all edges in U and W are undirected.

We prove the statement by induction on $n = |V(G)|$. The statement holds for $n = 3$ clearly. Assume that the statement holds for $n-1$ with $n \geq 4$ and we prove it holds for n. It is well-known that a chordal graph has a perfect elimination ordering, which is an ordering of the vertices such that, for each vertex v, the vertex v and the neighbors of v that occur after v in the order form a clique. Assume that $\{v_1, v_2, \ldots, v_n\}$ is a perfect ordering. By the inductive hypothesis, $M_G - v_1$ has a coincident cut X, say $V(G - v_1) = U \cup W$ such that the edges in U and W are undirected. According to the definition of X, we have either all edges between U and W are undirected or they have the same direction. Therefore, we divide two cases to discuss.

Case 1. All edges between U and W are undirected.

For any $u \in N_U(v_1)$ and $w \in N_W(v_1)$, since v_1, u, w form a clique and $M_G[v_1, u_w] \in \mathcal{C}_3$, we have either $v_1 \leftrightarrow u, w$, $v_1 \to u, w$ or $u, w \to v_1$. If the first case occurs, then there is nothing to prove. If the second one occurs, then, for any $u' \in N_U(v_1)$, we have $v \to u'$ since v_1, u, u' form a clique and $M_G[v_1, u, u'] \in \mathcal{C}_3$. Similarly, we have $v_1 \to w'$ for any $w' \in N_W(v_1)$. Therefore, all edges between v_1 and $U \cup W$ form the desired coincident cut. If the last one occurs, one can similarly verify that all edges between v_1 and $U \cup W$ form the desired coincident cut.

Case 2. All edges between U and W have the same direction, say $u \to v$ for any $u \in U$ and $w \in W$ with $u \sim v$ in G.

For any $u \in N_U(v_1)$ and $w \in N_W(v_1)$, since v_1, u, w form a clique and $M_G[v_1, u, w] \in \mathcal{C}_3$, we have either $v_1 \leftrightarrow u$ and $v_1 \to w$, or $u \to v_1$ and $v_1 \to w$. If the former occurs, then, for any $u' \in N_U(v_1)$, we have $v_1 \leftrightarrow u'$ since v_1, u, u' form a clique and $M_G[v_1, u, u'] \in \mathcal{C}_3$. Similarly, we have $v_1 \to w'$ for any $w' \in N_W(v_1)$. Therefore, all edges between $\{v_1\} \cup U$ and W form the desired coincident cut. If the latter occurs, one can similarly verify that all edges between U and $\{v_1\} \cup W$ form the desired coincident cut.

The proof is completed. $\square$

Remark 1. We prove Theorem 1 by investigating the structure of chordal graphs, and omit any reference to the spectral theory of complex unit gain graphs. In fact, Theorem 1, as well as Corollary 2, is immediate from Lemma 6 since all induced cycles of a chordal graph are triangles and all of the triangles are in $\mathcal{C}_3$ which are positive.

For nonnegative integers s, t, n with $n = s + t$, denote by $K_n[s, t]$ the mixed graph obtained from $K_s \cup K_t$ by adding all arcs from the vertices of K_s to those of K_t. We may assume that $K_n[n, 0] = K_n[0, n] = K_n$. From Theorem 1 it immediately follows that $K_n[s, t]$ is switching equivalent to K_n. In fact, we will show that $[K_n] = \{K_n[s, t] \mid s, t \geq 0, s + t = n\}$ and give a characterization of the graph set $[K_n]$.

Lemma 9. *Let M_{K_n} be a mixed graph on $n \geq 3$ vertices in which any mixed triangle belongs to $\mathcal{C}_3$. If M_{K_n} contains $K_3^{2,2}$, then $M_{K_n} \in \{K_n[s, t] \mid s \geq 2, t \geq 1, s + t = n\}$.*

Proof. Assume that $u, v, w \in V(M_{K_n})$ induce a $K_3^{2,2}$ with $u \to w$, $v \to w$ and $u \leftrightarrow v$. For any vertex $x \in V(M_{K_n}) \setminus \{u, v, w\}$ (if exists), we have either $x \leftrightarrow w$ or $x \to w$ since otherwise $M_{K_n}[u, w, x] \notin \mathcal{C}_3$. By noticing $M_{K_n}[u, x, w], M_{K_n}[v, x, w] \in \mathcal{C}_3$, one can easily verify that $v \to x$ and $u \to x$ if $x \leftrightarrow w$, and $x \leftrightarrow v$ and $u \leftrightarrow x$ if $x \to w$.

Denote by $V_1 = \{x \in V(M_{K_n}) \mid w \leftrightarrow x\} \cup \{w\}$ and $V_2 = \{x \in V(M_{K_n}) \mid w \leftarrow x\}$. It is clear that $u, v \in V_2$ and $V = V_1 \cup V_2$. For any two vertices $x_1, x_1' \in V_1 \setminus \{w\}$, we have $x_1 \leftrightarrow x_1'$ since $x_1, x_1' \leftrightarrow w$ and $M_G[x_1, x_1', w] \in \mathcal{C}_3$. Similarly, we have $x_2 \leftrightarrow x_2'$ for any $x_2, x_2' \in V_2$. Moreover, for any $x_1 \in V_1 \setminus \{w\}$ and $x_2 \in V_2$, we have $x_2 \to x_1$ since $x_1 \leftrightarrow w$, $x_2 \to w$ and $M_G[x_1, x_2, w] \in \mathcal{C}_3$. Thus, $M_{K_n} = K_n[s, t]$ where $s = |V_2| \geq 2$ and $t = |V_1| \geq 1$. $\square$

Similarly, we get the following result.

Lemma 10. *Let M_{K_n} be a mixed graph on $n \geq 3$ vertices in which any mixed triangular belongs to $\mathcal{C}_3$. If M_{K_n} contains $K_3^{2,3}$, then $M_{K_n} \in \{K_n[s, t] \mid s \geq 1, t \geq 2, s + t = n\}$.*

Proof. Assume that u, v, ω induce a $K_3^{2,3}$ with $u \leftarrow \omega$, $v \leftarrow \omega$ and $u \leftrightarrow v$. For any vertex $x \in V(M_{K_n}) \backslash \{u, v, \omega\}$ (if exists), we have either $x \leftrightarrow \omega$ or $x \leftarrow \omega$ since otherwise $M_{K_n}[u, w, x] \notin \mathcal{C}_3$. Note that $M_{K_n}[u, x, w], M_{K_n}[v, x, w] \in \mathcal{C}_3$. We have $x \to u$ and $x \to v$ if $\omega \leftrightarrow x$, and $x \leftrightarrow u$ and $x \leftrightarrow v$ if $\omega \to x$. Let $V_3 = \{x \in V(M_{K_n}) \mid \omega \leftrightarrow x\} \cup \{\omega\}$ and $V_4 = \{x \in V(M_{K_n}) \mid \omega \to x\}$.

Clearly, $V(M_{K_n}) = V_3 \cup V_4$, $u, v \in V_3$. Taking $x_3, x_3' \in V_3$ and $x_4, x_4' \in V_4$, we get $x_3 \leftrightarrow x_3'$ and $x_4 \leftrightarrow x_4'$, Therefore, V_3 and V_4 induce a clique, respectively, and $|V_3| \geq 1$, $|V_4| \geq 2$. Moreover, we also have $x_3 \to x_4$ for any $x_3 \in V_3, x_4 \in V_4$. Therefore, we get $M_{K_n} = K_n[s, t]$ with $s = |V_3| \geq 1$ and $t = |V_4| \geq 2$. $\square$

Lemmas 9 and 10 yield the following result.

Theorem 2. *Let M_{K_n} be a mixed graph with underlying graph K_n and $n \geq 3$. Then the following statements are equivalent:*

(i) *any mixed triangle of M_{K_n} belongs to $\mathcal{C}_3$;*
(ii) *$M_{K_n} \in \{K_n[s, t] \mid s, t \geq 0, s + t = n\}$;*
(iii) *$M_{K_n} \in [K_n]$.*

Proof. Firstly, assume that any triangle of M_{K_n} belongs to $\mathcal{C}_3$. Lemma 9 and Lemma 10 indicate that $M_{K_n} \in \{K_n[s, t] \mid s \geq 1, t \geq 1, s + t = n\}$ when M_G contains $K_3^{2,2}$ or $K_3^{2,3}$. If M_{K_n} contains neither $K_3^{2,2}$ nor $K_3^{2,3}$, then any mixed triangle of M_{K_n} is K_3, and thus $M_{K_n} = K_n = K_n[n, 0]$. Conversely, one can easily verify that any mixed triangle of $K_n[s, t]$ belongs to $\mathcal{C}_3$. Thus, $(i) \Leftrightarrow (ii)$.

Next we will show $[K_n] = \{K_n[s, t] \mid s, t \geq 0, s + t = n\}$. It is clear that $\{K_n[s, t] \mid s, t \geq 0, s + t = n\} \subseteq [K_n]$. It suffices to show that $[K_n] \subseteq \{K_n[s, t] \mid s, t \geq 0, s + t = n\}$. By the arguments above, it only needs to show that any mixed triangle in M_{K_n} belongs to $\mathcal{C}_3$ for any $M_{K_n} \in [K_n]$. Assume that $H(M_{K_n}) = [h_{j,k}]$ for a mixed graph $M_{K_n} \in [K_n]$. Since $M_{K_n} \in [K_n]$, there exists a diagonal matrix $D = diag(d_1, d_2, \ldots, d_n)$ with $d_j \in \{\pm\mathbf{i}, \pm 1\}$ such that $DH(M_{K_n})D^* = H(K_n)$. Therefore, for any $\{u, v, w\} \subseteq V(M_{K_n})$, we have

$$\begin{pmatrix} d_u & & \\ & d_v & \\ & & d_w \end{pmatrix} \begin{pmatrix} 0 & h_{uv} & \overline{h}_{wu} \\ \overline{h}_{uv} & 0 & h_{vw} \\ h_{wu} & \overline{h}_{vw} & 0 \end{pmatrix} \begin{pmatrix} \overline{d_u} & & \\ & \overline{d_v} & \\ & & \overline{d_w} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

It leads to $d_u h_{uv} \overline{d_v} = 1$, $d_v h_{vw} \overline{d_w} = 1$ and $d_w h_{wu} \overline{d_u} = 1$. Thus, we have $h_{uv} h_{vw} h_{wu} = 1$. It implies that either exactly one of h_{uv}, h_{vw}, h_{wu} equals to 1 or all of them equal to 1. If the former case happens, say $h_{uv} = 1$, then $\{h_{vw}, h_{wu}\} = \{\pm\mathbf{i}\}$, which means $M_{K_n}[u, v, w] = K_3^{2,2}$ or $K_3^{2,3}$. If the latter case happens, then $M_{K_n}[u, v, w] = K_3$. Therefore, $(ii) \Leftrightarrow (iii)$.

The proof is completed. $\square$

Now we give a simple application of Theorem 2 as follows.

Theorem 3. *Let M_G be a connected mixed graph on n vertices. Then $\lambda_{\min}(M_G) > -\sqrt{2}$ if and only if $M_G \in \{K_n[s,t] \mid s,t \geq 0, s+t = n\}$.*

Proof. Theorem 2 implies that the mixed graph $K_n[s,t]$ has the spectrum $\{n-1, [-1]^{n-1}\}$, and the sufficiency follows. Now we consider the necessity. Assume that M_G is a mixed graph on n vertices with $\lambda_{\min}(M_G) > -\sqrt{2}$. Since $\mathrm{Sp}(M_{P_3}) = \mathrm{Sp}(P_3) = \{\pm\sqrt{2}, 0\}$, the path P_3 cannot be an induced subgraph of G due to Corollary 1. Thus, we have $G = K_n$. Furthermore, since $\lambda_{\min}(M_G) > -\sqrt{2} > -\sqrt{3}$, Lemma 7 also implies that each triangle in M_G belongs to $\mathcal{C}_3$. Thus, we have $M_G \in \{K_n[s,t] \mid s,t \geq 0, s+t = n\}$ by Theorem 2. □

Theorem 3 gives the characterization of mixed graphs with $\lambda_{\min} > -\sqrt{2}$. In what follows, we will further determine the mixed graphs with $\lambda_{\min} \geq -\sqrt{2}$.

Lemma 11. *Let M_G be a connected mixed graph on n vertices. If $\lambda_{\min}(M_G) \geq -\sqrt{2}$, then G is $\{P_3\nabla K_1, (K_2 \cup K_1)\nabla K_1\}$-free.*

Proof. Suppose to the contrary that G contains induced H for $H \in \{P_3\nabla K_1, (K_2 \cup K_1)\nabla K_1\}$. Therefore, Corollary 1 means that $\lambda_4(M_H) \geq -\sqrt{2} > -\sqrt{3}$, and thus each mixed triangle of M_H belongs to $\mathcal{C}_3$. Note that H has no cycle with length greater than 3. Theorem 1 implies that $\lambda_4(M_H) = \lambda_4(H)$, which equals to $\lambda_4(P_3\nabla K_1) = -1.56 < -\sqrt{2}$ or $\lambda_4((K_2 \cup K_1)\nabla K_1) = -1.48 < -\sqrt{2}$, a contradiction. □

By Lemma 11, we get the following result.

Theorem 4. *Let M_G be a connected mixed graph on $n \geq 4$ vertices. Then $\lambda_{\min}(M_G) \geq -\sqrt{2}$ if and only if $M_{K_n} \in \{K_n[s,t] \mid s,t \geq 0, s+t = n\} \cup \mathcal{C}_4$.*

Proof. The sufficiency is immediate and we show the necessity in what follows. We divide two cases to discuss.

Case 1. G is P_3-free.

In this case, we have $G = K_n$. Since $\lambda_{\min}(M_G) \geq -\sqrt{2} > -\sqrt{3}$, any mixed triangle in M_G belongs to $\mathcal{C}_3$ by Lemma 7. Thus, Theorem 2 means $M_G \in \{K_n[s,t] \mid s,t \geq 0, s+t = n\}$.

Case 2. G is not P_3-free.

In this case, suppose that there exists $u, v, w \in V(G)$ such that $G[u,v,w] = P_3$ with $u \sim v$ and $v \sim w$. Note that $\lambda_4(P_4) \approx -1.618 < -\sqrt{2}$ and $\lambda_4(K_{1,3}) = -\sqrt{3} < -\sqrt{2}$. Corollary 1 implies that G is $\{P_4, K_{1,3}\}$-free, and thus the diameter $d(G) = 2$. Therefore, each vertex $y \in V(G) \setminus \{u,v,w\}$ of $V(G)$ is adjacent to at least one vertex of $\{u,v,w\}$. If y is adjacent to exactly one vertex of $\{u,v,w\}$, then G either contains an induced P_4 or $K_{1,3}$, which is impossible. If y is adjacent to all the vertices $\{u,v,w\}$, then $G[u,v,w,y] = P_3\nabla K_1$, which contradicts Lemma 11. Thus, y is adjacent to exactly two vertices of $\{u,v,w\}$. If $y \sim u, v$ or $y \sim v, w$, then $G[u,v,w,y] = (K_2 \cup K_1)\nabla K_1$, which contradicts Lemma 11. Thus, $y \sim u, w$, that is $G[u,v,w,y] = C_4$. Next, we claim that $n = 4$. Otherwise, there exists another vertex $y' \in V(G) \setminus \{u,v,w,y\}$. By regarding y' as y, we have $G[u,v,w,y'] = C_4$. Therefore, we have $G[u,v,y,y'] = K_{1,3}$ when $y \nsim y'$ and $G[u,v,y,y'] = (K_2 \cup K_1)\nabla K_1$ when $y \sim y'$, which are all impossible. Therefore, we have $G = C_4$, and thus $M_G \in \mathcal{C}_4$ by Fig. 1.

This completes the proof. □

In what follows, we characterize the mixed graph M_G with $\lambda_{\min}(M_G) > -\frac{1+\sqrt{5}}{2}$. We first find some structural constraints on the underlying graph G.

Lemma 12. *If M_G is a mixed graph with underlying graph $G = K_{m,n}$, then $\lambda_{\min}(M_G) \leq -\frac{1+\sqrt{5}}{2}$ except for $G \in \{K_2, K_{1,2}, K_{2,2}\}$.*

Proof. Assume $\lambda_{\min}(M_G) > -\frac{1+\sqrt{5}}{2}$, then G has no $K_{1,3}$ as an induced subgraph since $\lambda_3(K_{1,3}) = -\sqrt{3} < -\frac{1+\sqrt{5}}{2} \approx -1.618$. This leads to $G \in \{K_2, K_{1,2}, K_{2,2}\}$. □

By applying Theorem 1, we get the following result.

Lemma 13. *Let M_G be a connected mixed graph on n vertices. If $\lambda_{\min}(M_G) > -\frac{1+\sqrt{5}}{2}$ then G is $\{P_4, K_{1,3}, K_{2,3}, 2K_1\nabla K_{1,2} = K_1\nabla K_{2,2}, K_2\nabla 3K_1, K_2\nabla(K_2 \cup K_1), K_2\nabla K_{1,2}, 2K_1\nabla K_3\}$-free.*

Proof. By Corollary 2, we have $\lambda_4(M_{P_4}) = \lambda_4(P_4) = -\frac{1+\sqrt{5}}{2}$ for any $M_{P_4} \in \mathcal{M}_{P_4}$ and $\lambda_4(M_{K_{1,3}}) = \lambda_4(K_{1,3}) = -1.73 < -\frac{1+\sqrt{5}}{2}$ for any $M_{K_{1,3}} \in \mathcal{M}_{K_{1,3}}$. Thus, Corollary 1 implies that G is $\{P_4, K_{1,3}\}$-free. Suppose to the contrary that G contains an induced $K_{2,3}$. Corollary 1 indicates that $\lambda_5(M_{K_{2,3}}) > -\frac{1+\sqrt{5}}{2}$, which contradicts Lemma 12.

Suppose to the contrary that G contains an induced $K_1\nabla K_{2,2}$ labeled as Fig. 2. Since $M = M_{K_1\nabla K_{2,2}}$ has smallest eigenvalue greater than $-\frac{\sqrt{5}+1}{2}$, Lemma 7 implies all mixed triangles of M belong to $\mathcal{C}_3$ and Lemma 8 implies all quadrangles of M belong to $\mathcal{C}_4$. If the mixed induced quadrangle $M_{K_{2,2}}$ is equal to C_4^3, then we have either $u_1 \to v$ or $v \to u_2$ since

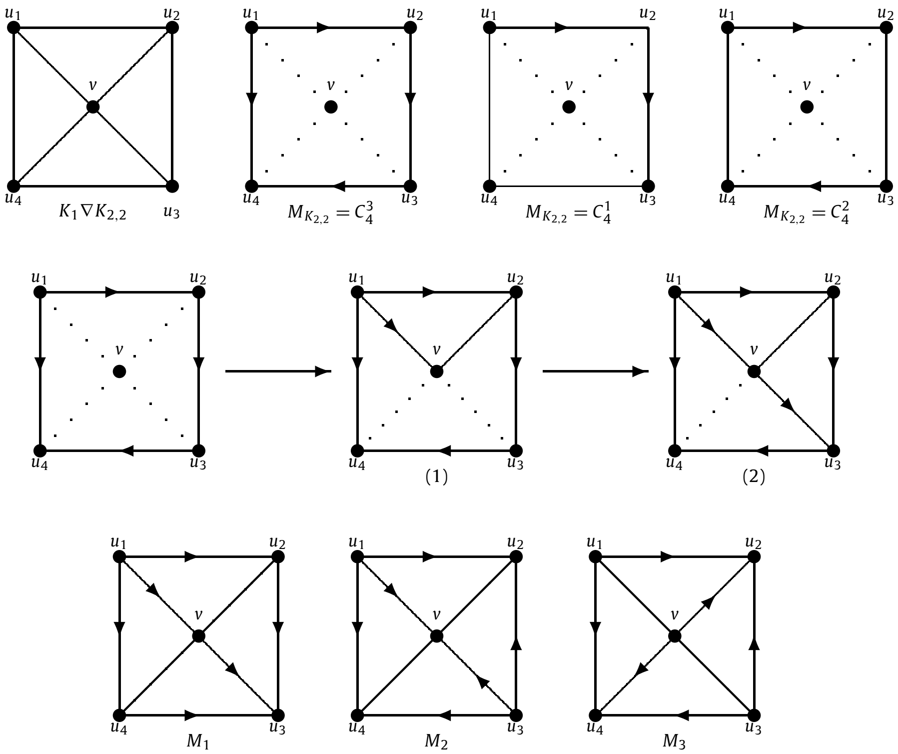


Fig. 2. The graphs used in Lemma 13.

$M[u_1, v, u_2] \in \mathcal{C}_3$. It leads to that $u_1 \to v$ and thus $v \leftrightarrow u_2$ since otherwise $v \to u_2$ and $M[v, u_2, u_3] \notin \mathcal{C}_3$ (see Fig. 2(1)). Since $v \leftrightarrow u_2$, $u_2 \to u_3$ and $M[v, u_2, u_3] \in \mathcal{C}_3$, we have $v \to u_3$ (see Fig. 2(2)). However, $M[v, u_3, u_4]$ cannot belong to $\mathcal{C}_3$, a contradiction. Similarly, if the mixed induced quadrangle $M_{K_{2,2}}$ is equal to C_4^1 or C_4^2, then $M \in \{M_1, M_2, M_3\}$ whose smallest eigenvalues are all $-2 < -\frac{\sqrt{5}+1}{2}$, a contradiction.

Suppose to the contrary that G contains an induced subgraph H in $\{K_2\nabla 3K_1, K_2\nabla(K_2 \cup K_1), K_2\nabla K_{1,2}, 2K_1\nabla K_3\}$. Therefore, M_G contains a mixed induced graph M_H with order 5. Corollary 1 indicates that $\lambda_m(M_H) \geq \lambda_{\min}(M_G) > -\frac{1+\sqrt{5}}{2}$. Thus, each mixed triangle in M_H belongs to $\mathcal{C}_3$. Note that H contains no cycle of length greater than 3. Theorem 1 implies that $\lambda_m(M_H) = \lambda_m(H)$. It leads to a contradiction since $\lambda_5(K_2\nabla 3K_1) = -2$, $\lambda_5(K_2\nabla(K_2 \cup K_1)) = -1.68$, $\lambda_5(K_2\nabla K_{1,2}) = -1.65$ and $\lambda_5(2K_1\nabla K_3) = -1.65$ which are all smaller than $-\frac{1+\sqrt{5}}{2}$. $\square$

From Lemma 13, we determine the underlying graphs of M_G with smallest eigenvalue greater than $-\frac{1+\sqrt{5}}{2}$.

Lemma 14. *Let M_G be a connected mixed graph on n vertices. If $\lambda_{\min}(M_G) > -\frac{1+\sqrt{5}}{2}$, then G belongs to*

$$\left\{K_{2,2}, K_1\nabla K_{1,2}, 2K_2\nabla 2K_1, (K_2 \cup K_1)\nabla 2K_1\right\} \bigcup \{(K_s \cup K_t)\nabla K_1 \mid s, t \geq 0, s+t = n-1\}.$$

Proof. We may assume that $n \geq 2$ since there is nothing to prove when $n = 1$. From Lemma 13, we have G is P_4-free and thus $G = X\nabla Y$ with $|X|, |Y| \geq 1$ due to Lemma 3. If both X and Y have no edge, then $G = K_{m,n}$ and thus $G \in \{K_2, K_{1,2}, K_{2,2}\}$ due to Lemma 12, where both $K_2 = (K_1 \cup K_0)\nabla K_1$ and $K_{1,2} = (K_1 \cup K_1)\nabla K_1$ have the form $(K_s \cup K_t)\nabla K_1$. Now we may assume that one of X and Y contains K_2, say X. Therefore, Lemma 13 implies that Y is $\{3K_1, K_2 \cup K_1, K_{1,2}\}$-free and thus $Y \in \{2K_1, K_s \mid s \geq 1\}$ due to Lemma 4. If $Y = K_s$ with $s \geq 2$, then Lemma 13 implies that X is $\{3K_1, K_2 \cup K_1, K_{1,2}\}$-free. Thus, Lemma 4 means that $X = K_r$ with $r \geq 2$ since X contains K_2. Therefore, $G = K_n = (K_{n-1} \cup K_0)\nabla K_1$ with $n \geq 4$. If $Y = 2K_1$, then Lemma 13 indicates that X is $\{3K_1, K_{1,2}, K_3\}$-free. Hence, $X \in \{2K_2, K_2 \cup K_1, K_2\}$ due to Lemma 4, and thus $G \in \{2K_2\nabla 2K_1, (K_2 \cup K_1)\nabla 2K_1, K_2\nabla 2K_1 = K_1\nabla K_{1,2}\}$.

In what follows, we consider the case of $Y = K_1$, that is $G = X\nabla K_1$. Since G is $K_{1,3}$-free according to Lemma 13, we have X is $\{3K_1\}$-free and X has at most two connected components. Suppose that X has two connected components, say X_1 and X_2 with $|X_1|, |X_2| \geq 1$. Then both X_1 and X_2 are P_3-free since otherwise X has an induced $3K_1$, and so X_1 and X_2 are complete graphs. Therefore, $G = (K_s \cup K_t)\nabla K_1$ with $s+t = n-1$ and $s, t \geq 1$. Next we may assume that X is connected. Since X is P_4-free, we have $X = X_1\nabla Y_1$ with $|X_1|, |Y_1| \geq 1$ from Lemma 3. If both X_1 and Y_1 have no edges, then X is a bipartite graph and so $X \in \{K_2, K_{1,2}, K_{2,2}\}$ by Lemma 12. Note that Lemma 13 means that G is $K_1\nabla K_{2,2}$-free. Thus, $G \in \{K_3, K_1\nabla K_{1,2}\}$. Now we may assume that X_1 contains a K_2. Then Y_1 is $\{3K_1, K_2 \cup K_1, K_{1,2}\}$-free by Lemma 13. Hence,

$Y_1 \in \{K_s, 2K_1\}$ by Lemma 4. If $Y_1 = K_s$ with $s \geq 2$, then X_1 is $\{3K_1, K_2 \cup K_1, K_{1,2}\}$-free by Lemma 13. By Lemma 4, we have $X_1 = K_t (t \geq 2)$ since X_1 has an edge. Note that $G = X \nabla K_1 = (X_1 \nabla Y_1)\nabla K_1$. Therefore, $G = (K_s \nabla K_t)\nabla K_1 = K_n$ for $n \geq 5$. If $Y_1 = 2K_1$, then $G = X_1 \nabla 2K_1 \nabla K_1 = X_1 \nabla K_{1,2}$. Lemma 13 indicates that X_1 is $\{2K_1, K_2\}$-free, and thus $X_1 = K_1$. Therefore, $G = K_1 \nabla K_{1,2}$. If $Y_1 = K_1$, then $G = X_1 \nabla K_1 \nabla K_1 = X_1 \nabla K_2$. Therefore X_1 is $\{3K_1, K_2 \cup K_1, K_{1,2}\}$-free by Lemma 13. Lemma 4 indicates that $X_1 = K_s (s \geq 2)$ since X_1 has an edge. Thus, $G = K_s \nabla K_2 = (K_{n-1} \nabla K_0)\nabla K_1$ with $n \geq 4$.

The proof is completed. $\square$

In what follows, we detect all mixed graphs whose smallest eigenvalues are greater $-\frac{1+\sqrt{5}}{2}$ by considering one by one all the possible underlying graphs.

Lemma 15. *Let M_G be a mixed graph with $G = (K_2 \cup K_1)\nabla 2K_1$. If any mixed triangle of M_G belongs to $\mathcal{C}_3$ and any induced mixed quadrangle of M_G belongs to $\mathcal{C}_4$, then $M_G \in \{H_1, \dots, H_9\}$ shown in the Appendix.*

Proof. Let $V(G) = \{v_1, v_2, \dots, v_5\}$ (see the Appendix). Clearly, M_G has two induced quadrangles $M_G[v_1, v_2, v_3, v_4]$ and $M_G[v_1, v_5, v_3, v_4]$. Note that any induced mixed quadrangle belongs to $\mathcal{C}_4$. We divide four cases to discuss.

Case 1. One of them is C_4^3.

In this case, we may assume $M_G[v_1, v_2, v_3, v_4] = C_4^3$. Clearly, there are two different orientations of the mixed cycle $M_G[v_1, v_2, v_3, v_4]$: $v_1 \to v_2 \to v_3 \to v_4$ and $v_1 \to v_4$; $v_2 \to v_3 \to v_4 \to v_1$ and $v_2 \to v_1$. If the former happens, then $M_G[v_1, v_5, v_3, v_4] = C_4^3$, and thus $v_1 \to v_5$, $v_5 \to v_3$ and $v_2 \leftrightarrow v_5$ since any induced mixed triangle belongs to $\mathcal{C}_3$. It yields that $M_G = H_1$. If the latter happens, then $M_G[v_1, v_5, v_3, v_4] = C_4^1$ or C_4^3. Therefore, one can easily verify that $M_G = H_2$ when $M_G[v_1, v_5, v_3, v_4] = C_4^1$, and $M_G = H_3$ when $M_G[v_1, v_5, v_3, v_4] = C_4^3$.

Case 2. $M_G[v_1, v_2, v_3, v_4] = M_G[v_1, v_5, v_3, v_4] = C_4^2$.

In this case, there are also two different orientations of $M_G[v_1, v_2, v_3, v_4]$: $v_1 \to v_2$, $v_2 \leftrightarrow v_3$, $v_3 \to v_4$, $v_4 \leftrightarrow v_1$; $v_1 \leftrightarrow v_2$, $v_2 \to v_3$, $v_3 \leftrightarrow v_4$ and $v_4 \to v_1$. Therefore, one can easily verify that $M_G = H_4$ when the former happens and $M_G = H_5$ when the latter happens by noticing that any mixed triangle belongs to $\mathcal{C}_3$.

Case 3. $M_G[v_1, v_2, v_3, v_4] = M_G[v_1, v_5, v_3, v_4] = C_4^1$.

In this case, there are three different orientations of $M_G[v_1, v_2, v_3, v_4]$: $v_1 \to v_2$, $v_2 \to v_3$, $v_3 \leftrightarrow v_4$ and $v_4 \leftrightarrow v_1$; $v_1 \leftrightarrow v_2$, $v_2 \to v_3$, $v_3 \to v_4$ and $v_4 \leftrightarrow v_1$; $v_1 \leftrightarrow v_2$, $v_2 \leftrightarrow v_3$, $v_3 \to v_4$ and $v_4 \to v_1$. One can easily verify that $M_G = H_6$ when the first case happens, $M_G = H_7$ when the second case happens, and $M_G = H_8$ when the third case happens.

Case 4. $M_G[v_1, v_2, v_3, v_4] = C_4^1$ and $M_G[v_1, v_5, v_3, v_4] = C_4^2$.

In this case, there are three different orientations of $M_G[v_1, v_2, v_3, v_4]$: $v_1 \to v_2$, $v_2 \to v_3$, $v_3 \leftrightarrow v_4$ and $v_4 \leftrightarrow v_1$; $v_1 \leftrightarrow v_2$, $v_2 \to v_3$, $v_3 \to v_4$ and $v_4 \leftrightarrow v_1$; $v_1 \leftrightarrow v_2$, $v_2 \leftrightarrow v_3$, $v_3 \to v_4$ and $v_4 \to v_1$. If the first or the third case happens, then $M_G[v_1, v_5, v_3, v_4]$ cannot be C_4^2, which is impossible. If the second case happens, then $M_G = H_9$.

This completes the proof. $\square$

As similar to Lemma 15, we present the following result but omit the tautological proof.

Lemma 16. *Let M_G be a mixed graph with $G = 2K_2 \nabla 2K_1$. If any mixed triangle of M_G belongs to $\mathcal{C}_3$ and any induced mixed quadrangle belongs to $\mathcal{C}_4$, then $M_G \in \{H_{10}, \dots, H_{20}\}$ shown in the Appendix.*

The *coalescence* $M \bullet_{u,v} M'$ of two mixed graphs M and M' is obtained from $M \cup M'$ by identifying a vertex u of M with a vertex v of M'.

Lemma 17. *Let G be a connected graph with a cut vertex v such that $G - v = G_1 \cup G_2$ with $V_1 = V(G_1)$ and $V_2 = V(G_2)$. If $G_1^+ = G[V_1 \cup \{v\}]$ and $G_2^+ = G[V_2 \cup \{v\}]$, then $[G] = \{M \bullet_{v,v} M' \mid M \in [M_{G_1^+}], M' \in [G_2^+]\}$.*

Proof. It is clear that $G = G_1^+ \bullet_{v,v} G_2^+$. For any $M \bullet M'$ with $M \in [M_{G_1^+}]$ and $M' \in [G_2^+]$, there exist diagonal matrices D_1 and D_2 with diagonal entries in $\{\pm 1, \pm \mathbf{i}\}$ such that $D_1 H(M) D_1^* = H(G_1^+)$ and $D_2 H(M') D_2 = H(G_2^+)$. Note that the v-th diagonal entries of D_1 and D_2 satisfy $D_2(v) = \epsilon D_1(v)$ for some $\epsilon \in \{\pm 1, \pm \mathbf{i}\}$. Let D be the diagonal matrix indexed by $V(G)$ such that the diagonal entries are $D(v_1) = D_1(v_1)$ for $v_1 \in V_1 \cup \{v\}$ and $D(v_2) = \epsilon D_2(v_2)$ for $v_2 \in V_2$. Therefore, one can easily verify that $DH(M \bullet_{v,v} M')D^* = H(G_1^+ \bullet_{v,v} G_2^+) = H(G)$, and thus $M \bullet_{v,v} M' \in [G]$.

Conversely, for any $M_G \in [G]$, there exists diagonal matrix D such that $DH(M_G)D^* = H(G)$. Note that $M_G = M \bullet_{v,v} M'$ where $M = M_G[V_1 \cup \{v\}]$ and $M' = M_G[V_2 \cup \{v\}]$. Let D_1 and D_2 be the diagonal matrices indexed by $V_1 \cup \{v\}$ and $V_2 \cup \{v\}$ respectively such that the diagonal entries are $D_1(v_1) = D(v_1)$ for any $v_1 \in V_1 \cup \{v\}$ and $D_2(v_2) = D(v_2)$ for any $v_2 \in V_2 \cup \{v\}$. Therefore, one can easily verify that $D_1 H(M) D_1^* = H(G_1^+)$ and $D_2 H(M') D_2^* = H(G_2^+)$, and thus $M \in [G_1^+]$ and $M' \in [G_2^+]$. $\square$

Now we are ready to present our main result.

Theorem 5. *Let M_G be a connected mixed graph on n vertices. Then $\lambda_{\min} > -\frac{1+\sqrt{5}}{2}$ if and only if $M_G \in \mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{H}_3 \cup \mathcal{H}_4$, where*

$$\begin{cases} \mathcal{H}_1 = \{C_4^1, C_4^2, C_4^3, H_1, H_2, \dots, H_{27}\}, \\ \mathcal{H}_2 = \{M \bullet_{u,v} M' \mid u \in V(M), v \in V(M'), M \in [K_3], M' \in [K_3] \cup [K_4]\}, \\ \mathcal{H}_3 = [K_n] = \{K_n[s,t] \mid s,t \geq 0, s+t=n\}, \\ \mathcal{H}_4 = \{M \bullet_{u,v} M' \mid u \in V(M), v \in V(M'), M \in [K_2], M' \in [K_{n-1}]\}. \end{cases}$$

Proof. To prove the sufficiency, it only needs to show that each graph in $\mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{H}_3 \cup \mathcal{H}_4$ has smallest eigenvalue greater than $-\frac{1+\sqrt{5}}{2}$. By immediate calculations, the smallest eigenvalues of C_4^1, C_4^2, C_4^3 are all $-\sqrt{2} > -\frac{1+\sqrt{5}}{2}$ and the smallest eigenvalues of $H_1, H_2, \dots, H_{27}$ are all $-1.56 > -\frac{1+\sqrt{5}}{2}$ (see the Appendix). For any $M \in [K_3]$ and $M' \in [K_3]$, Lemma 17 implies that $\mathrm{Sp}(M \bullet_{u,v} M') = \mathrm{Sp}(K_3 \bullet_{u,v} K_3)$. Thus, we have $\mathrm{Sp}(M \bullet_{u,v} M') = \{2.56, 1, [-1]^2, -1.56\}$ by immediate calculations. Similarly, if $M \in [K_3]$ and $M' \in [K_4]$, we have $\mathrm{Sp}(M \bullet_{u,v} M') = \{3.26, 1.34, [-1]^3, -1.60\}$. Theorem 2 implies that $K_n[s,t]$ has smallest eigenvalue -1. For any $M_G \in \mathcal{H}_4$, Lemma 17 implies that $\mathrm{Sp}(M_G) = \mathrm{Sp}(K_{n-1} \bullet_{u,v} K_2)$, whose smallest eigenvalue is the smallest root of $\varphi(x) = x^3 + (3-n)x^2 + (1-n)x - 1 = 0$. Note that $\varphi(-1) = 0$, $\varphi(-\frac{1+\sqrt{5}}{2}) = 1-n < 0$ for $n \geq 2$. The smallest root of $\varphi(x)$ is greater than $-\frac{1+\sqrt{5}}{2}$ by the image of $\varphi(x)$, and thus $\lambda_{\min}(M_G) > -\frac{1+\sqrt{5}}{2}$.

In what follows, we show the necessity. Since $\lambda_{\min}(M_G) > -\frac{1+\sqrt{5}}{2}$, Lemmas 7 and 8 indicate that any mixed triangle of M_G belongs to $\mathcal{C}_3$ and any mixed induced quadrangle of M_G belongs to $\mathcal{C}_4$. From Lemma 14, the underlying graph G belongs to

$$\{K_{2,2}, K_1\nabla K_{1,2}, 2K_2\nabla 2K_1, (K_2\cup K_1)\nabla 2K_1\} \bigcup \{(K_s \cup K_t)\nabla K_1 \mid s+t = n-1\}.$$

If $G = K_{2,2}$, then $M_G \in \{C_4^1, C_4^2, C_4^3\} \subseteq \mathcal{H}_1$ due to Lemma 8. If $G = K_1\nabla K_{1,2}$, then G contains no induced cycle with length greater than 3. Thus, Theorem 1 implies that $M_G \in [K_1\nabla K_{1,2}] = \{H_{21}, \dots, H_{27}\} \subseteq \mathcal{H}_1$. If $G = 2K_2\nabla 2K_1$ or $(K_2 \cup K_1)\nabla 2K_1$, then $M_G \in \{H_1, \dots, H_{20}\} \subseteq \mathcal{H}_1$ due to Lemmas 15 and 16.

If $G = (K_s \cup K_t)\nabla K_1$ with $s = 0$ or $t = 0$, then $G = K_n$. Since any mixed triangle of M_G belongs to $\mathcal{C}_3$, Theorem 2 means that $M_G = M_{K_n} \in \{K_n[s,t] \mid s,t \geq 0, s+t=n\} = [K_n] = \mathcal{H}_3$. Now we suppose $G = (K_s \cup K_t)\nabla K_1$ with $s,t \geq 1$ and $s \geq t$. Note that G contains no induced cycle with length greater than 3, Theorem 1 indicates that $M_G \in [G]$ and thus $\mathrm{Sp}(M_G) = \mathrm{Sp}(G)$. Note that $[(K_s \cup K_t)\nabla K_1] = \{M \bullet_{u,v} M' \mid M \in [K_{s+1}], M' \in [K_{t+1}]\}$ due to Lemma 17. Assume that $\pi\colon V(G) = V_1 \cup \{v\} \cup V_2$ is the partition such that $G[V_1 \cup \{v\}] = K_{s+1}$ and $G[V_2 \cup \{v\}] = K_{t+1}$. The Hermitian matrix of G is

$$H(G) = \begin{pmatrix} J_s - I_s & \mathbf{1}_s & \mathbf{0}_{s\times t} \\ \mathbf{1}_s^T & \mathbf{0}_{t\times s} & \mathbf{1}_t^T \\ 0 & \mathbf{1} & J_s - I_s \end{pmatrix},$$

where J, I, $\mathbf{1}$ and $\mathbf{0}$ are respectively the all-one matrix, identity matrix, all-one vector and zero matrix with the corresponding sizes. Therefore, Lemma 2 indicates that π is an equitable partition with quotient matrix

$$H_\pi = \begin{pmatrix} s-1 & 1 & 0 \\ s & 0 & t \\ 0 & 1 & t-1 \end{pmatrix}.$$

Assume that $V_1 = \{v_1, v_2, \dots, v_s\}$ and $V_2 = \{u_1, u_2, \dots, u_t\}$. For $1 \leq j \leq s$ and $1 \leq k \leq t$, let $\delta_{1,j} \in \mathbb{R}^s$ be the vector indexed by V_1 such that $\delta_{1,j}(v_1) = 1$, $\delta_{1,j}(v_j) = -1$ and $\delta_{1,j}(v_{j'}) = 0$ for $j' \notin \{1, j\}$ and let $\delta_{2,k}$ be the vector indexed by V_2 such that $\delta_{2,k}(u_1) = 1$, $\delta_{2,k}(u_k) = -1$ and $\delta_{2,k}(u_{k'}) = 0$ for $k' \notin \{1,k\}$. It is easy to see that $H(G)\delta_{1,j} = -\delta_{1,j}$ and $H(G)\delta_{2,k} = -\delta_{2,k}$ for any j and k, and thus H has an eigenvalue -1 with multiplicity at least $s+t-2 = n-3$. Lemma 2 implies that the other three eigenvalues of G are just the roots $\epsilon_1 \geq \epsilon_2 \geq \epsilon_3$ of the function $f(x) = det(xI - B_\pi) = x^3 + (2-t-s)x^2 + (st - 2t - 2s + 1)x - s - t + 2st$, and thus $\epsilon_3 = \lambda_{\min}(G) > -\frac{1+\sqrt{5}}{2}$. It is clear that $f(0) = st - s - t \geq 0$. Note that $\epsilon_1 > 0$. By the image of the function $f(x)$, we have $f(-\frac{1+\sqrt{5}}{2}) < 0$. If $t \geq 3$ then

$$f(-\frac{1+\sqrt{5}}{2}) = \frac{3-\sqrt{5}}{2}(st - s - t) + \frac{1-\sqrt{5}}{2} \geq \frac{3-\sqrt{5}}{2}s + \frac{1-\sqrt{5}}{2} \geq 5 - 2\sqrt{5} > 0,$$

a contradiction. Thus, we have $t \leq 2$. If $t = 2$ then $f(-\frac{1+\sqrt{5}}{2}) = \frac{3-\sqrt{5}}{2}s - \frac{5-\sqrt{5}}{2} < 0$. It leads to $s < \frac{5+\sqrt{5}}{2} \approx 3.62$. Thus, we have $s = 2$ or 3 since $s \geq t = 2$. It means $M_G \in [(K_2 \cup K_2)\nabla K_1] \cup [(K_3 \cup K_2)\nabla K_1] = \{M \bullet_{u,v} M' \mid M \in [K_3], M' \in [K_3] \cup [K_4]\} = \mathcal{H}_2$. If $t = 1$ then $f(-\frac{1+\sqrt{5}}{2}) = -1 < 0$ always holds. Thus, $s \geq t = 1$ and $M_G \in [(K_s \cup K_1)\nabla K_1] = \{M \bullet_{u,v} M' \mid M \in [K_2], M' \in [K_{n-1}]\} = \mathcal{H}_4$.

This completes the proof. □

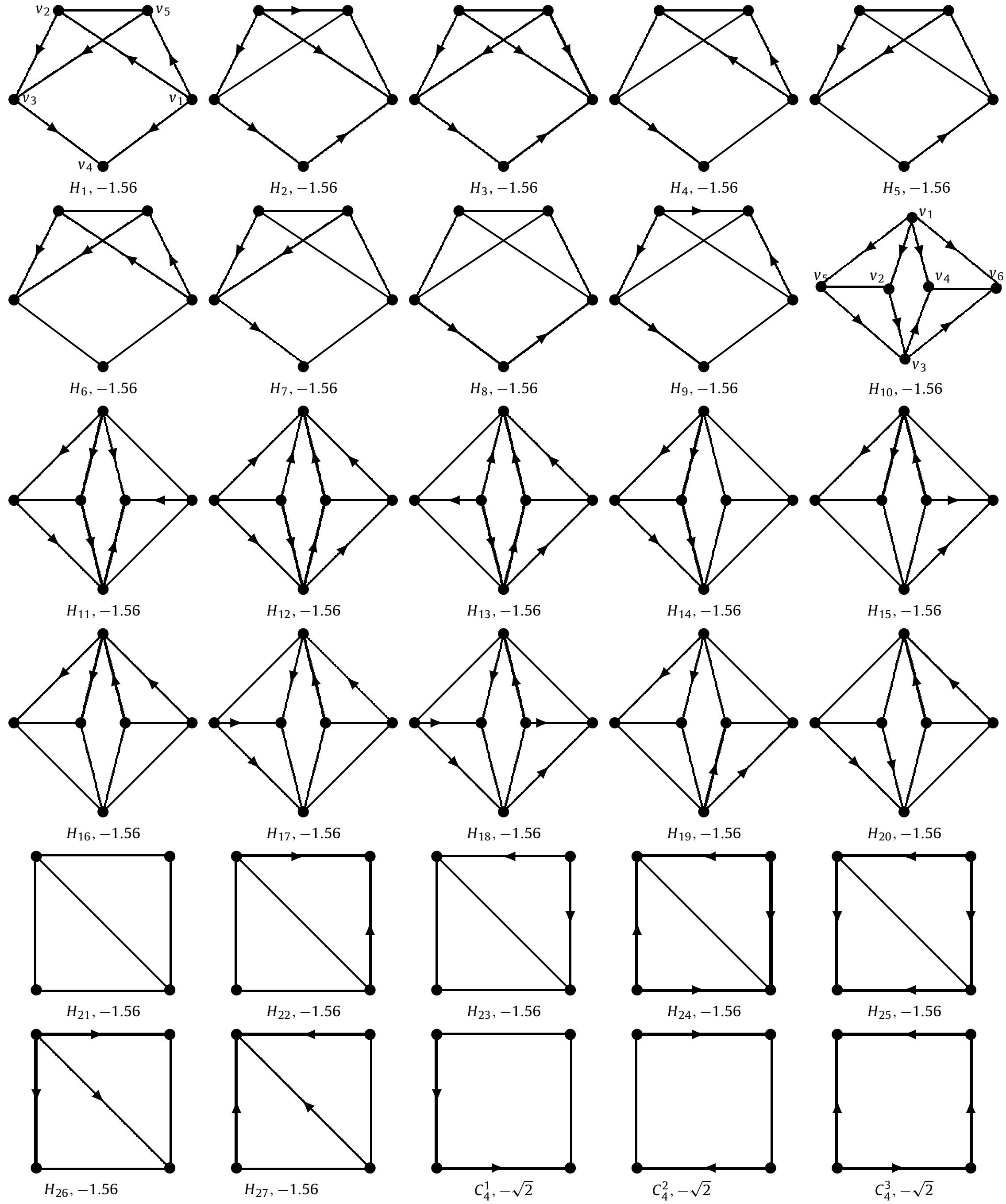
Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

This work is supported by NSFC (Nos. 12001544, 12061074, 11971274), Natural Science Foundation of Hunan Province (No. 2021JJ40707) and the China Postdoctoral Science Foundation (No. 2019M661398). The authors are so grateful to the referees for their valuable comments and corrections which improve the presentation of the paper. Especially, he/she points out the results in [2], which simplify some proofs of this paper.

Appendix A. The mixed graphs $\mathcal{C}_4$ and $H_1, \ldots, H_{27}$ with their smallest eigenvalues



$H_1, -1.56$ $H_2, -1.56$ $H_3, -1.56$ $H_4, -1.56$ $H_5, -1.56$

$H_6, -1.56$ $H_7, -1.56$ $H_8, -1.56$ $H_9, -1.56$ $H_{10}, -1.56$

$H_{11}, -1.56$ $H_{12}, -1.56$ $H_{13}, -1.56$ $H_{14}, -1.56$ $H_{15}, -1.56$

$H_{16}, -1.56$ $H_{17}, -1.56$ $H_{18}, -1.56$ $H_{19}, -1.56$ $H_{20}, -1.56$

$H_{21}, -1.56$ $H_{22}, -1.56$ $H_{23}, -1.56$ $H_{24}, -1.56$ $H_{25}, -1.56$

$H_{26}, -1.56$ $H_{27}, -1.56$ $C_4^1, -\sqrt{2}$ $C_4^2, -\sqrt{2}$ $C_4^3, -\sqrt{2}$

References

[1] A. Abdollahi, E.R. van Dam, M. Jazaeri, Distance-regular Cayley graphs with least eigenvalue -2, Des. Codes Cryptogr. 84 (2017) 73–85.
[2] A. Alazemi, M. Anđelić, F. Belardo, M. Brunetti, C.M. da Fonseca, Line and subdivision graphs determined by $\mathbb{T}_4$-gain graphs, Mathematics 7 (926) (2019), https://doi.org/10.3390/math7100926.
[3] B. Bollobás, V. Nikiforov, Graphs and Hermitian matrices: eigenvalue interlacing, Discrete Math. 289 (2004) 119–127.
[4] A.E. Brouwer, A. Neumaier, The graphs with spectral radius between 2 and $\sqrt{2+\sqrt{5}}$, Linear Algebra Appl. 114/115 (1989) 273–276.
[5] D.M. Cvetković, M. Doob, I. Gutman, On graphs whose spectral radius does not exceed $\sqrt{2+\sqrt{5}}$, Ars Comb. 14 (1982) 225–239.
[6] D.M. Cvetković, P. Rowlinson, S. Simić, Spectral Generalizations of Line Graphs: On Graphs with Least Eigenvalue -2, Cambridge University Press, Cambridge, 2004.
[7] D.M. Cvetković, P. Rowlinson, S. Simić, Graphs with least eigenvalue -2: ten years on, Linear Algebra Appl. 484 (2015) 504–539.
[8] C.D. Godsil, G. Royle, Algebraic Graph Theory, Springer-Verlag, Berlin, 2001.
[9] K. Guo, B. Mohar, Hermitian adjacency matrix of digraphs and mixed graphs, J. Graph Theory 85 (1) (2016) 217–248.
[10] K. Guo, B. Mohar, Digraphs with Hermitian spectral radius below 2 and their cospectrality with paths, Discrete Math. 340 (2017) 2616–2632.
[11] A.J. Hoffman, On limit points of spectral radii of nonnegative symmetric integral matrices, in: Y. Alavi, D.R. Lick, A.T. White (Eds.), Graph Theory and Application, Proceedings of a Conference at Western Michigan University, 1972, in: Lecture Notes in Math., vol. 303, Springer, Berlin, 1972, pp. 165–172.
[12] A. Hoffman, On graphs whose least eigenvalue exceeds $-1-\sqrt{2}$, Linear Algebra Appl. 16 (1977) 153–165.
[13] J.H. Koolen, J. Yang, Q. Yang, On graphs with smallest eigenvalue at least -3 and their lattices, Adv. Math. 338 (2018) 847–864.
[14] S. Kubota, T. Taniguchi, K. Yoshino, On graphs with the smallest eigenvalue at least $-1-\sqrt{2}$, part III, Ars Math. Contemp. 17 (2019) 555–579.
[15] J. Li, X. Li, Hermitian-adjacency matrices and Hermitian energies of mixed graphs, Linear Algebra Appl. 466 (2015) 182–207.
[16] A. Munemasa, Y. Sano, T. Taniguchi, Fat Hoffman graphs with smallest eigenvalue at least $-1-\tau$, Ars Math. Contemp. 7 (2014) 247–262.
[17] D. Seinsche, On a property of the class of n-colorable graphs, J. Comb. Theory, Ser. B 16 (1974) 191–193.
[18] J.H. Smith, Some properties of the spectrum of a graph, in: Combinatorial Structures and Their Applications (Proc. Calgary Internat. Conf., Calgary, Alta., 1969), Gordon and Breach, New York, 1970, pp. 403–406.
[19] T. Taniguchi, On graphs with the smallest eigenvalue at least $-1-\sqrt{2}$, part I, Ars Math. Contemp. 1 (2008) 81–98.
[20] T. Taniguchi, On graphs with the smallest eigenvalue at least $-1-\sqrt{2}$, part II, Ars Math. Contemp. 5 (2012) 243–258.
[21] J.F. Wang, F. Belardo, Q.X. Huang, On graphs whose Laplacian index does not exceed 4.5, Linear Algebra Appl. 438 (2013) 1541–1550.
[22] J.F. Wang, Q.X. Huang, F. Belardo, E.M. Li Marzi, On graphs whose signless Laplacian index does not exceed 4.5, Linear Algebra Appl. 431 (2009) 162–178.
[23] J.F. Wang, X.Y. Lei, W. Wei, X.B. Luo, S.C. Li, On the eccentricity matrix of graphs and its applications to the boiling point of hydrocarbons, Chemom. Intell. Lab. Syst. 207 (2020) 104173.
[24] J.F. Wang, M. Lu, L. Lu, F. Belardo, Spectral properties of the eccentricity matrix of graphs, Discrete Appl. Math. 279 (2020) 168–177.
[25] Y. Wang, B. Yuan, On graphs whose orientations are determined by their Hermitian spectra, Electron. J. Comb. 27 (3) (2020) P3.55.
[26] P. Wissing, E.R. van Dam, The negative tetrahedron and the first infinite family of connected digraphs that are strongly determined by the Hermitian spectrum, J. Comb. Theory, Ser. A 173 (2020) 105232.
[27] R. Woo, A. Neumaier, On graphs whose spectral radius is bounded by $\frac{3}{2}\sqrt{2}$, Graphs Comb. 23 (2007) 713–726.
[28] B. Yuan, Y. Wang, S. Gong, Y. Qiao, On mixed graphs whose Hermitian spectral radii are at most 2, Graphs Comb. 36 (2020) 1573–1584.