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Spectral determinations and eccentricity matrix of graphs

Jianfeng Wang[a,*], Mei Lu[b], Maurizio Brunetti[c], Lu Lu[d], Xueyi Huang[e]

[a] *School of Mathematics and Statistics, Shandong University of Technology, Zibo 255049, China*
[b] *Department of Mathematical Sciences, TsingHua University, Beijing 100084, China*
[c] *Department of Mathematics and Applications, University of Naples 'Federico II', Italy*
[d] *School of Mathematics and Statistics, Central South University, Changsha 410083, China*
[e] *School of Mathematics, East China University of Science and Technology, Shanghai 200237, China*

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ABSTRACT

Let G be a connected graph on n vertices. For a vertex $u \in G$, the eccentricity of u is defined as $\varepsilon(u) = \max\{d(u,v) \mid v \in V(G)\}$, where $d(u,v)$ denotes the distance between u and v. The eccentricity matrix $\mathcal{E}(G) = (\epsilon_{uv})$, where

$$\epsilon_{uv} := \begin{cases} d(u,v) & \text{if } d(u,v) = \min\{\varepsilon(u), \varepsilon(v)\}, \\ 0 & \text{otherwise,} \end{cases}$$

has been firstly introduced in Chemical Graph Theory. In literature, it is also known as the D_{MAX}-matrix. Graphs with the diameter equal to the radius are called self-centered graphs. Two non-isomorphic graphs are said to be M-cospectral with respect to a given matrix M if they have the same M-eigenvalues. In this paper, we show that, when $n \to \infty$, the fractions of non-isomorphic cospectral graphs with respect to the adjacency and the eccentricity matrix

* Corresponding author.
E-mail addresses: jfwang@sdut.edu.cn (J. Wang), lumei@tsinghua.edu.cn (M. Lu), maurizio.brunetti@unina.it (M. Brunetti), lulugdmath@163.com (L. Lu), huangxymath@163.com (X. Huang).

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behave like those only concerning the self-centered graphs with diameter two. Secondly, we prove that a graph G has just two distinct $\mathcal{E}$-eigenvalues if and only if G is an r-antipodal graph. Thirdly, we obtain many pairs of $\mathcal{E}$-cospectral graphs by using strong and lexicographic products. Finally we formulate some problems waiting to be solved in order to build up a spectral theory based on the eccentricity matrix.

1. Introduction

Our main source for notation and terminology is [13]. Even when not explicitly stated, all graphs in this paper are assumed to be simple, undirected and connected. Let $G = (V(G), E(G))$ be such a graph with order $|V(G)| = n$ and size $|E(G)| = m$. Let $M = M(G)$ be a corresponding *graph matrix* defined in a prescribed way. The *M-polynomial* of G is defined as $\phi_M(G, \lambda) = \det(\lambda I - M)$, where I is the identity matrix. The *M-spectrum* of G, denoted by $\mathrm{sp}_M(G)$, is the multiset of the M-eigenvalues. Among the most studied graph matrices, there are the adjacency matrix $A(G)$, the Laplacian matrix $L(G)$, the signless Laplacian matrix $Q(G)$ and the distance matrix $\mathcal{D}(G)$. Noteworthily, a newer graph matrix, the Hermitian adjacency matrix, was introduced by Guo and Mohar [29] and Liu and Li [38] independently.

Non-isomorphic graphs with the same M-spectrum are called *M-cospectral*. A property $\mathcal{P}$ is said to be *M-cospectral invariant* if for every pair G, H of graphs such that $\mathcal{P}$ holds for G and $\mathrm{sp}_M(G) = \mathrm{sp}_M(H)$, then $\mathcal{P}$ holds for H as well. An *M-cospectral mate* of G is a graph H which is M-cospectral to G. If no M-cospectral mates of G exist, we say that G is determined by its M-spectrum, or equivalently that G is a DMS-*graph*. The problem 'which graphs are determined by their spectrum?' originates from Chemistry: in fact, it was formulated in 1956 by Günthard and Primas [28] in the context of Hückel's theory. The same problem has often been considered in the more general setting of spectral graph theory. Yet, the investigations progressed very slowly along the past fifty years. For additional remarks on this topic we refer the readers to the excellent surveys [18,19]. Theorem 1.1 below states a quite famous result in this field.

Theorem 1.1. [42] *Almost all trees are A-cospectral.*

Schwenk, who proved Theorem 1.1, also proposed the following still unsolved conjecture.

Conjecture 1. [42] *Almost all graphs are A-cospectral.*

Table 1
Fractions of M-cospectral graphs-I.

n	3	4	5	6	7	8	9	10	11
A	0	0	0.059	0.064	0.105	0.139	0.186	0.213	0.211
$A\&\overline{A}$	0	0	0	0	0.038	0.094	0.160	0.201	0.208
L	0	0	0	0.026	0.125	0.143	0.155	0.118	0.090
Q	0	0.182	0.118	0.103	0.098	0.097	0.069	0.053	0.038

Parallel conjectures can be stated with respect to other graph matrices. If fact, many researchers have tried to grasp whether almost all graphs are M-cospectral for suitable $M \neq A(G)$.

Notably, Schwenk pointed that Theorem 1.1 offers no guidance for his Conjecture 1. On the other hand, based on the evidence provided by computer data available in [27,30] and summed up in Table 1, the authors of [18,30] gave credit to the still widely open Conjecture 2, formally proposed by Haemers in a talk at the 21$^{\text{th}}$ ILAS Conference in 2017 (see [31]). Note that, by definition, the generalized spectrum of a graph G is the A-spectrum together with the adjacency spectrum of the complement $\overline{G}$, i.e., the graph with $V(\overline{G}) = V(G)$ such that v_i and v_j are adjacent in $\bar{G}$ if and only if they are not adjacent in G.

Conjecture 2. [31] *Almost all graphs are determined by the adjacency (or Laplacian, or generalized) spectrum.*

As it is usual, P_n, C_n, S_n and K_n will denote the *path*, the *cycle*, the *star* and the *complete graph* of order n, respectively. The *union* of two disjoint graphs G_1 and G_2, denoted by $G_1 \cup G_2$, is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. The *complete product* (or *join*) of two graphs, denoted by $G_1 \nabla G_2$, is obtained from $G_1 \cup G_2$ by adding all edges between G_1 and G_2.

The other sections of this paper are organized as follows. In Section 2, we introduce the distance-like matrix of graphs known as the *eccentricity matrix* $\mathcal{E}(G)$ of a graph G. The $\mathcal{E}$-spectra of graphs with order at most 9 arouses the expectation that the fraction of $\mathcal{E}$-cospectral graphs tends to 1 as $n \to \infty$. In Section 3, we discuss how *self-centered graphs* (we shall often refer to them as SC-graphs), *distance-regular graphs* and *antipodal graphs* are related, giving for such third family of graphs an $\mathcal{E}$-eigenvalue characterization. In Section 4, we describe some techniques to get $\mathcal{E}$-cospectral pairs. Finally, in Section 5, we list some open problems in $\mathcal{E}$-spectral theory and some existing partial answers given to them.

2. A new perspective for the spectral determination of graphs

2.1. A distance-related matrix: the eccentricity matrix

We now define the eccentricity matrix and investigate the role it plays in Conjectures 1 and 2.

For a graph G, the *distance* $d_G(v,w)$ between two vertices v and w is the length of the shortest (v,w)-path connecting them. Clearly, $d_G(v,v)=0$. Let $\mathcal{D}(G)=(d_{uv})$ be the *distance matrix* of G, where $d_{uv}=d_G(u,v)$. Related to the distance matrix, the distance (signless) Laplacian matrix $\mathcal{D}^L$ ($\mathcal{D}^Q$) was introduced by Aouchiche and Hansen [2]. The *eccentricity* $\varepsilon_G(u)$ of the vertex $u \in V(G)$ is given by $\varepsilon_G(u)=\max\{d(u,v) \mid v \in V(G)\}$ (the subscript G will be omitted if the context makes it clear). The numbers

$$\operatorname{rad}(G) := \min\{\varepsilon(u)|u \in V(G)\} \qquad \text{and} \qquad \operatorname{diam}(G) := \max\{\varepsilon(u)|u \in V(G)\}$$

give the *radius* and the *diameter* of G respectively. They are known to satisfy the relations

$$\operatorname{rad}(G) \le \operatorname{diam}(G) \le 2\operatorname{rad}(G). \tag{1}$$

As in [45], the generic element ϵ_{uv} of the eccentricity matrix $\mathcal{E}(G)$ of G is defined as follows:

$$\epsilon_{uv} = \begin{cases} d(u,v) & \text{if } d(u,v)=\min\{\varepsilon(u),\varepsilon(v)\}; \\ 0 & \text{otherwise.} \end{cases}$$

By comparing the definitions, it turns out that $\mathcal{E}(G)$ is equal to the D_{MAX}-matrix introduced by Randić in [41] as a tool for Chemical Graph Theory. Since the importance of vertex-eccentricity is not limited to applications to chemistry, the author asserted that such a matrix might open new directions of exploration in other branches of graph theory as well. The matrix $\mathcal{E}(G)$ is constructed from $\mathcal{D}(G)$ by only keeping the largest distances for each row and each column, whereas the remaining entries become null. That is why $\mathcal{E}(G)$ can be interpreted as the opposite of the adjacency matrix, which is instead constructed from $\mathcal{D}(G)$ by keeping only distances equal to 1 on each row and each column. From this point of view, $A(G)$ and $\mathcal{E}(G)$ are extremal among all possible distance-like matrices.

Since $\mathcal{E}(G)$ is symmetric, the $\mathcal{E}$-eigenvalues are real. Therefore, the $\mathcal{E}$-spectrum of G can be written as

$$\operatorname{sp}_{\mathcal{E}}(G) = \begin{Bmatrix} \xi_1 & \xi_2 & \cdots & \xi_k \\ m_1 & m_2 & \cdots & m_k \end{Bmatrix},$$

where $\xi_1 > \xi_2 > \cdots > \xi_k$ are the distinct $\mathcal{E}$-eigenvalues, and m_i is the multiplicity of the eigenvalue ξ_i for each $i=1,2,\ldots,k$.

The first three rows of Table 2 are taken from [2]. To fill out the fourth one, we first used Sagemath, a computer program for generating graphs, and secondly we have calculated with Mathematica the $\mathcal{E}$-spectrum of all connected graphs up to 9 vertices. In contrast to the behavior of the other 7 type of matrices considered in Tables 1 and 2, the data about $\mathcal{E}$-matrices reveal that the proportion of $\mathcal{E}$-cospectral graphs is nearly 50% for the graphs with 7, 8 or 9 vertices, arousing the expectation that the fraction of $\mathcal{E}$-cospectral graphs tends to 1 as $n \to \infty$.

Table 2
Fractions of M-cospectral graphs-II.

n	3	4	5	6	7	8	9	10
$\mathcal{D}$	0	0	0	0	0.0258	0.0592	0.0960	0.1186
$\mathcal{D}^L$	0	0	0	0	0.0670	0.0504	0.0758	0.0672
$\mathcal{D}^Q$	0	0	0.0952	0.0536	0.0445	0.0407	0.0313	0.073
$\mathcal{E}$	0	0	0.0952	0.2143	0.4576	0.4464	0.4953	—

Proposition 2.1. *Let G be a connected graph of order n, and let $\overline{G}$ be its complement.*

(i) *If* $\mathrm{diam}(G) = 2$ *and* $\Delta(G) < n - 1$, *then*

$$A(\overline{G}) = \mathcal{D}(G) + I_n - J_n = \frac{1}{2}\mathcal{E}(G). \tag{2}$$

(ii) *If* $\mathrm{diam}(G) > 3$, *then* $A(G) = \frac{1}{2}\mathcal{E}(\overline{G})$.

Proof. Clearly, (i) holds. For any two vertices $u, v \in V(G)$, if $uv \notin E(G)$, then $d_{\overline{G}}(u, v) = 1$. If $uv \in E(G)$, there exists at least one vertex w such that both uw and vw are not in $E(G)$, otherwise $\mathrm{diam}(G) < 3$, which is not the case. This implies that $d_{\overline{G}}(u, v) = 2$. Having proved that $\mathrm{diam}(G) = 2$, by part (i) we have $\mathcal{E}(\overline{G}) = 2A(G)$. $\square$

Lemma 2.2. *Almost all connected graphs have radius and diameter equal to* 2.

Proof. By the principle of inclusion-exclusion, the probability that at least one vertex is adjacent to all the others is at most $n/2^{n-1}$, a number going to 0 as $n \to \infty$. It follows that almost all graphs with order n have maximum degree $\Delta(G) < n - 1$. Since it is known that almost all connected graphs have diameter 2 (see [6]), by $\mathrm{rad}(G) \leq \mathrm{diam}(G)$ we can infer that almost all graphs with order n have diameter 2 and maximum degree $\Delta(G) < n - 1$. $\square$

Lemma 2.2 and Equality (2) yield to the following important fact.

Proposition 2.3. *Almost all connected graphs satisfy the equality* $\mathcal{E}(G) = 2A(\overline{G})$.

Proposition 2.3 establishes a strong relationship between the spectral properties of the eccentricity matrix and Conjecture 1, whereas Propositions 2.1 and 2.4 somehow relate the matrices A, $\mathcal{D}$ and $\mathcal{E}$. All those results give us a sufficient motivation to investigate and construct $\mathcal{E}$-cospectral graphs later on.

As already observed, the A-matrix and $\mathcal{E}$-matrix can be both obtained from the $\mathcal{D}$-matrix, moreover $A(K_n) = \mathcal{D}(K_n) = \mathcal{E}(K_n)$.

Proposition 2.4. *Let G be a connected graph with order n. Then*

(i) $\mathcal{E}(G) = A(G)$ *if and only if* $G = K_n$;

(ii) $\mathcal{E}(G) = \mathcal{D}(G)$ *if and only if* $G \cong K_r \nabla \overline{K_{n-r}}$ $(1 \leq r \leq n-1)$.

Proof. For (i), sufficiency is clear. From $\mathcal{E}(G) = A(G)$, it follows that all non-zero entries of $\mathcal{E}(G)$ are equal to 1. This implies that $\varepsilon_G(v) = 1$ for any vertex $v \in V(G)$. This only happens when $G = K_n$.

We next show (ii). To avoid trivial cases we assume $n > 1$. Let $G \cong K_r \nabla \overline{K_{n-r}}$. The case $r = n-1$ is immediate since $K_{n-1} \nabla \overline{K_1} \cong K_n$. Suppose now $r < n-1$. Set $\mathcal{D}(K_r \nabla \overline{K_{n-r}}) = (d_{ij})_{n\times n}$ and $\mathcal{E}(K_r \nabla \overline{K_{n-r}}) = (\epsilon_{ij})_{n\times n}$. Assuming that the first r vertices of the ordered set $\{v_1, \ldots, v_n\} = V(G)$ belong to $V(K_r)$, we have

$$\varepsilon(v_i) = \begin{cases} 1 & \text{if } i \leq r; \\ 2 & \text{otherwise.} \end{cases}$$

By analyzing the three cases $i < j \leq r$; $i \leq r < j$, and $r < i < j$, it is now easy to check that for all $i \neq j$, $d(v_i, v_j) = \min\{\varepsilon(v_i), \varepsilon(v_j)\}$, which means that $\mathcal{E}(G) = \mathcal{D}(G)$.

Conversely, suppose $\mathcal{E}(G) = \mathcal{D}(G)$. Since G is connected, for any vertex $u \in V(G)$ we get a vertex v adjacent to u. Therefore, we get $\epsilon_{uv} = d_{uv} = 1$ and thus either $\varepsilon(u) = 1$ or $\varepsilon(v) = 1$. Let $V_1 = \{v \in V(G) \mid \varepsilon(v) = 1\}$ and $V_2 = V(G) \setminus V_1$. If V_2 is empty, we get $G = K_n = K_{n-1} \nabla \overline{K_1}$, and there is nothing else to prove.

If V_2 is non-empty, we consider the subgraph H induced by V_2.

By definition, each vertex of V_1 is adjacent to all other vertices. Thus, $G = K_r \nabla H$, where $r = |V_1|$. To complete the proof, we only need to show that the edge set of H is empty. This fact depends on the relation $d_{uv} = \epsilon_{uv} > 1$ holding for all pairs u and v of distinct vertices in V_2. $\square$

For sake of completeness, we include here the $\mathcal{E}$-spectra of paths and cycles computed by two authors of this paper elsewhere.

Proposition 2.5. [44] *Let* P_n *be a path of order* n.

(i) *If* $n = 1, 2, 3$, *then*

$$\mathrm{sp}_{\mathcal{E}}(P_1) = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}, \ \mathrm{sp}_{\mathcal{E}}(P_2) = \begin{Bmatrix} 1 & -1 \\ 1 & 1 \end{Bmatrix}, \ \mathrm{sp}_{\mathcal{E}}(P_3) = \begin{Bmatrix} 1+\sqrt{3} & 1-\sqrt{3} & -2 \\ 1 & 1 & 1 \end{Bmatrix}.$$

(ii) *If* $n = 2k$ $(k \geq 2)$, *then*

$$\mathrm{sp}_{\mathcal{E}}(P_{2k}) = \begin{Bmatrix} \frac{(6k-3)+\sqrt{a}}{6} & \frac{(6k-3)-\sqrt{a}}{6} & \frac{-(6k-3)+\sqrt{a}}{6} & \frac{-(6k-3)-\sqrt{a}}{6} & 0 \\ 1 & 1 & 1 & 1 & 2k-4 \end{Bmatrix},$$

where $a = (6k-3)(14k^2 - 20k + 9)$.

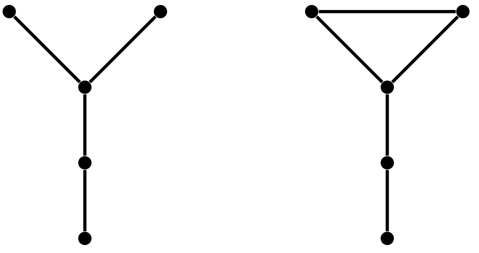

Fig. 1. The smallest pair of $\mathcal{E}$-cospectral graphs.

(iii) *If* $n = 2k+1$ $(k \geq 2)$, *then*

$$\operatorname{sp}_{\mathcal{E}}(P_{2k+1}) = \begin{Bmatrix} \frac{6k+\sqrt{b}}{6} & \frac{6k-\sqrt{b}}{6} & \frac{-6k+\sqrt{c}}{6} & \frac{-6k-\sqrt{c}}{6} & 0 \\ 1 & 1 & 1 & 1 & 2k-3 \end{Bmatrix},$$

where $b = 6k(14k^2+3k+1)$ *and* $c = 6k(14k^2-9k+1)$.

Proposition 2.6. [45] *Let* C_n *be a cycle of order* n.

(i) *If* $n = 2t$, *then*

$$\operatorname{sp}_{\mathcal{E}}(C_{2t}) = \begin{Bmatrix} t & -t \\ t & t \end{Bmatrix}.$$

(ii) *If* $n = 2t+1$, *then the* $\mathcal{E}$*-eigenvalues of* C_{2t+1} *are*

$$\xi_i = 2t\cos\frac{2\pi i}{2t+1} \quad (i = 1, \cdots, 2t+1).$$

We end this subsection by observing that $n = 5$ is the smallest order for a pair of graphs to be $\mathcal{E}$-cospectral. Such 'minimal' pair of graphs is depicted in Fig. 1.

2.2. Self-centered graphs and spectral determinations

Self-centered graphs had been studied extensively until the 1990s. Before giving their definition, we first recall some other distance-related concepts.

A *diametrical* (resp., *central*) vertex of G is a vertex u whose eccentricity is equal to $\operatorname{diam}(G)$ (resp., $\operatorname{rad}(G)$).

A connected graph G is said to be an *even graph* if each vertex u of G has a unique vertex $\bar{u}$ such that $d_G(u, \bar{u}) = \operatorname{diam}(G)$ (see [25]). It immediately follows that all vertices of an even graph are diametrical. In literature, even graphs are also known as *diametrical graphs* and *self-centered unique eccentric point graphs* (see, for instance [39,40]).

The center of a graph G, denoted by $C(G)$, is the set of vertices with minimum eccentricity. A connected graph G is said to be a *self-centered graph*[1] (or *SC-graph* for short) if $|C(G)| = |V(G)|$ (equivalently, if $\mathrm{rad}(G) = \mathrm{diam}(G)$). In other words, the first inequality in (1) is an equality if and only if G is self-centered.

It is immediately seen that even graphs are a special kind of SC-graphs. The even cycles and the cubes are examples of even graphs. For $n \geq 3$, the complete graph K_n and the regular complete k-partite graph $K_{n,\dots,n}$ are examples of non-even SC-graphs.

SC-graphs were discovered and independently studied by Akiya, Ando and Avis in 1981 [1]. Through $\mathcal{E}$-matrices, we have a nice characterization of them. In fact, a graph G is an SC-graph if and only if the non-zero elements of $\mathcal{E}(G)$ are equal. In Subsection 3.1 we shall give more examples involving antipodal graphs and distance-regular graphs.

It is well-known that the center of a graph/network is one of the central concepts in location theory. In many cases, an ideal situation would be that resources can be placed at any location. This can be done precisely when a graph is self-centered [35]. SC-graphs have been extensively studied (see [9,10,12,33] and the survey paper [11]). For instance, it is known that these graphs are necessarily 2-connected; moreover, for any finite group $\mathcal{G}$ there exists an SC-graph whose automorphism group is isomorphic to the group $\mathcal{G}$ [37].

In order to establish connections between SC-graphs and Conjectures 1 and 2, we could rephrase Lemma 2.2 by saying that almost all graphs are SC-graphs with diameter 2. Once this is done, the following two propositions are immediate from Lemma 2.2 and Proposition 2.3.

Proposition 2.7. *The following statements are equivalent.*

i) Conjecture 1 holds;

ii) almost all SC-graphs with diameter 2 *are A-cospectral;*

iii) almost all SC-graphs with diameter 2 *are $\mathcal{E}$-cospectral.*

Proposition 2.8. *The following statements are equivalent.*

i) Conjecture 2 holds;

ii) almost all SC-graphs with diameter two are determined by their A-spectrum;

iii) almost all SC-graphs with diameter two are determined by their $\mathcal{E}$-spectrum.

Propositions 2.7 and 2.8 say that both the Schwenk Conjecture and Haemers Conjecture could possibly be proved or disproved just by restricting the attention to SC-graphs with diameter 2. This is precisely what we do by filling the last two rows of Table 3, where the symbol $\#_M$ denotes the fraction of M-cospectral graphs among all SC-graphs with diameter 2.

Numbers in each column of Table 3 are all sufficiently close to empirically endorse the equivalence of Conditions i)-iii) in Propositions 2.7 and 2.8.

[1] It is also named as *diametral graph* by Fiol [21].

Table 3
Fractions of M-cospectral graphs-III.

n	3	4	5	6	7	8	9
A	0	0	0.0588	0.0641	0.1054	0.1395	0.1858
$\#_A$	0	0	0	0	0.0971	0.1373	0.1941
$\#_\mathcal{E}$	0	0	0	0	0.0826	0.1379	0.1921

Table 4
SC-graphs with small order.

n	3	4	5	6	7	8
SCG	1	2	5	28	221	3151
RSCG	1	2	2	5	4	16
DRG	1	2	2	4	2	5

3. Some families of SC-graphs

3.1. Distance-regular and antipodal graphs

Distance-regular graphs were introduced by Biggs in 1969. For any $v \in V(G)$ and any $i \geq 0$, we set $G_i(v) := \{u \in V(G) \,|\, d(u,v) = i\}$.

A k-regular connected graph G with diameter d is called *distance-regular with intersection array*

$$\{b_0 = k, b_1, \ldots, b_{d-1}; c_1 = 1, c_2, \ldots, c_d\}, \tag{3}$$

if for every pair of vertices $u, v \in V(G)$ at distance $i = d_G(u,v)$, there are precisely c_i neighbors of v in $G_{i-1}(u)$ and b_i neighbors of v in $G_{i+1}(u)$.

Distance-regular graphs play an important role in algebraic combinatorics because of their applications to design theory, coding theory, finite and Euclidean geometry, and group theory (see [7,8]). We just quote two among the most striking results on distance-regular graphs accomplished in the last two decades. Van Dam and Koolen provided in [20] the first known family of non-vertex-transitive distance-regular graphs with unbounded diameter. More recently, Bang, Dubickas, Koolen and Moulton [3] proved that there are only finitely many distance-regular graphs of fixed valency greater than 2.

We explicitly note that in a distance-regular graph all vertices share the same eccentricity. Hence, each distance-regular graph is a SC-graph.

In Table 4, we exhibit the number of SC-graphs (SCG), regular SC-graphs (RSCG) and distance-regular graphs (DRG) with order at most 8. The proportion of DRG-graphs among all SC-graphs is tiny, whereas it is by far larger with respect to the regular SC-graphs.

We then consider the following problem.

Problem 1. Under what conditions are regular SC-graphs also distance-regular graphs?

In order to formulate an answer, we need some further terminology and notation. Given any connected graph G with diameter d, for every non-negative integer $h \leq d$ the *distance-h graph* $\mathcal{G}_h$ is the graph on $V(G)$ where two vertices are adjacent whenever they are at distance h in G. The A-matrix of $\mathcal{G}_h$, denoted by $\mathcal{A}_h$, is usually referred as the *distance-h matrix* of G. Let $\lambda_0 > \lambda_1 > \cdots > \lambda_k$ be the distinct A-eigenvalues of G, and

$$\pi_\ell = \prod_{j=0, j\neq\ell}^{k} |\lambda_\ell - \lambda_j|. \tag{4}$$

As it is well-known, $k \leq d$ (see, for instance, Biggs [4, Corollary 2.7]). Then, we say that G is extremal when it has the maximum possible number of distinct A-eigenvalues, i.e., when $k = d$.

Proposition 3.1. [23] *A self-centered graph G with diameter d is distance-regular if and only if G is extremal and its distance-d matrix $\mathcal{A}_d$ is a polynomial of degree d in the adjacency matrix $A(G)$.*

Fiol and Garriga also obtained in [22] the following spectral characterization of regular graphs.

Proposition 3.2. [22, Theorem 4.4] *A regular (SC-graph) G with order n is distance-regular if and only if, for every $u \in V(G)$,*

$$|G_d(u)| = \frac{n}{\pi_0^2 \sum_{i=0}^{k} \frac{1}{m_i \pi_i^2}},$$

where m_i is the multiplicity of λ_i and π_i is defined in (4) $(0 \leq i \leq k)$.

We now turn our attention to antipodal graphs. A graph G of diameter d is called *antipodal* if there exists a partition of the vertex set into classes (called the *fibers* of G) with the property that two distinct vertices are in the same class if and only if they are at distance d. Each set in such partition has the form $\{u\} \cup G_d(u)$ (see, for instance, [26, Ch. 11.5]). If all fibers have the same cardinality, say r, we say that G is an r-antipodal graph. Clearly, the complete graph K_n is a n-antipodal graph with diameter 1.

Observe that r-antipodal graphs ($r \geq 2$) are SC-graphs, and even graphs are 2-antipodal graphs. In Subsection 3.2 we shall have the opportunity to recap some intriguing results on distance-regular and antipodal graphs. Among many others, they are collected in [17,21,24].

Fig. 2 summarizes how several types of graphs are related. Hooked arrows represent inclusions; the dotted line emphasizes that not all distance-regular graphs are antipodal graphs.

3.2. Eigenvalue characterization of antipodal graphs

In this subsection, we give an eigenvalue characterization of antipodal graphs. Such characterization will be useful to construct $\mathcal{E}$-cospectral graphs in Section 4.

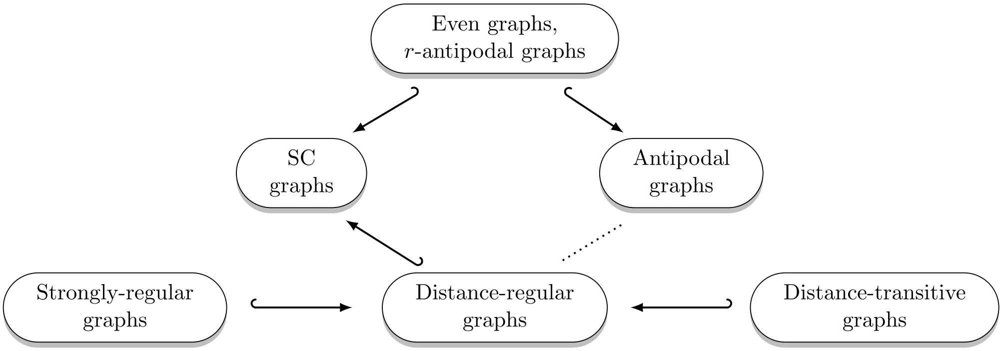

Fig. 2. Several types of graphs: a visual comparison.

Proverbially, the topic that the graphs with few distinct eigenvalues has been paid much attention. For more details see the Ph.D. thesis, due to van Dam [16]. It is quite well-known that if the M-spectrum of a connected graph G just contains two distinct eigenvalues for at least one $M \in \{A, Q, L, \mathcal{D}\}$, then G is a complete graph. Surprisingly, the situation for $M = \mathcal{E}$ is quite different, as we are about to show.

Lemma 3.3 *(Cauchy's Interlacing Theorem). Let R be a real symmetric $n \times n$ matrix and let S be a principal submatrix of R of order m. Then, for $i = 1, 2, \ldots, m$,*

$$\lambda_{n-m+i}(R) \leq \lambda_i(S) \leq \lambda_i(R),$$

where $\lambda_1(R) \geq \lambda_2(R) \geq \cdots \geq \lambda_n(R)$ and $\lambda_1(S) \geq \lambda_2(S) \geq \cdots \geq \lambda_m(S)$ are the eigenvalues of R and the eigenvalues of S respectively.

An (u, v)-path of G is called a *diametrical path* if its length is $d_G(u, v) = \mathrm{diam}(G)$.

Lemma 3.4. *Let G be a connected graph with order n and $\mathcal{E}(G)$ being irreducible. Then G has two distinct $\mathcal{E}$-eigenvalues if and only if $\mathcal{E}(G) = J_n - I_n$, i.e., G is the complete graph K_n.*

Proof. If $G = K_n$, then $\mathcal{E}(G) = \mathcal{E}(K_n) = J_n - I_n = A(K_n)$, which is irreducible. Its spectrum just contain $n - 1$ with multiplicity 1 and -1 with multiplicity $n - 1$.

Suppose now $\mathcal{E}(G)$ irreducible with just two distinct eigenvalues. It will suffice to show that $\mathrm{diam}(G) = 1$. In fact, if this is the case G is necessarily equal to K_n. Let P_{uv} be a diametrical path with diametrical vertices u and v. By contradiction we assume that there exists a vertex $w \in V(P_{uv}) \setminus \{u, v\}$. From Perron-Frobenius Theorem it follows that $\xi_1 \in \mathrm{sp}_{\mathcal{E}}(G)$ is an algebraically simple $\mathcal{E}$-eigenvalue. According to the parity of $N_{uv} := |V(P_{uv})|$ we distinguish the following cases:

Case 1: $N_{uv} = 2k + 1$. We show that $\mathcal{E}(G)$ contains a principal submatrix of order 3 with 3 different eigenvalues. This fact contradicts to Lemma 3.3.

If $d_{P_{uv}}(u,w) = d_{P_{uv}}(w,v)$, then $d_G(u,w) = d_G(w,v) = k$. If $\varepsilon_G(w) = d_G(w,v)$. Therefore, the principal submatrix indexed by the vertices u, v, w is similar (if not equal) to

$$M_1 = \begin{pmatrix} 0 & 2k & k \\ 2k & 0 & k \\ k & k & 0 \end{pmatrix}$$

which has three distinct eigenvalues; namely $(1+\sqrt{3})k, (1-\sqrt{3})k$, and $-2k$.

If instead $\varepsilon_G(w) > d_G(w,v)$, the principal submatrix indexed by u, v, w is similar to

$$M_2 = \begin{pmatrix} 0 & 2k & 0 \\ 2k & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Such matrix also has three distinct eigenvalues; namely, $2k$, 0, and $-2k$.

Suppose finally $d_{P_{uv}}(u,w) \neq d_{P_{uv}}(w,v)$. Without loss of generality, we can assume

$$d_G(w,v) = a > d_G(w,u).$$

In this case, the principal submatrix indexed by the vertices u, v, w is similar to

$$M_3 = \begin{pmatrix} 0 & 2k & 0 \\ 2k & 0 & a \\ 0 & a & 0 \end{pmatrix}, \qquad \text{where } k < a < 2k.$$

The matrix M_3 too has three distinct eigenvalues; namely $\sqrt{4k^2+a^2}, 0$, and $-\sqrt{4k^2+a^2}$.

Case 2: $N_{uv} = 2k$. Arguing as for Case 1, the principal submatrix corresponding to the rows & columns indexed by u, v, w always has three eigenvalues, contradicting Lemma 3.3. □

Lemma 3.5. *Let G be a connected graph with order n and diameter $d \geq 2$. The eccentrity matrix $\mathcal{E}(G)$ is reducible and has two distinct $\mathcal{E}$-eigenvalues if and only if there exists an integer $r \geq 2$ such that $r|n$ and, after possibly re-labeling the vertex set,*

$$\mathcal{E}(G) = dI_{\frac{n}{r}} \otimes (J_r - I_r),$$

where $\otimes$ denotes the Kronecker product. Moreover, the two distinct $\mathcal{E}$-eigenvalues are equal to the following pair of integers:

$$\xi_1 = d(r-1) \qquad \text{and} \qquad \xi_2 = -d. \tag{5}$$

Proof. The proof of sufficiency is short and easy. Recalling that, for every pair (X, Y) of $n \times n$ matrices,

$$\lambda \in \operatorname{sp}(X) \quad \text{and} \quad \mu \in \operatorname{sp}(Y) \implies \lambda\mu \in \operatorname{sp}(X \otimes Y),$$

the spectrum of the Kronecker product $\mathcal{E}(G) = dI_{\frac{n}{r}} \otimes (J_r - I_r)$ just contains the two eigenvalues defined in (5).

We now prove necessity by assuming that $\mathcal{E}(G)$ is reducible, and $\xi_1 > \xi_2$ are its only two distinct eigenvalues. Since $\mathrm{Tr}(\mathcal{E}(G)) = 0$, surely $\xi_1 > 0 > \xi_2$. After possibly re-labeling the vertex set, $\mathcal{E}(G)$ becomes a block diagonal matrix. More specifically,

$$\mathcal{E}(G) = \begin{pmatrix} X_{n_1} & O & O & \cdots & O & O \\ O & X_{n_2} & O & \cdots & O & O \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & O & \cdots & X_{n_{s-1}} & O \\ O & O & O & \cdots & O & X_{n_s} \end{pmatrix}, \tag{6}$$

where X_{n_i} $(1 \leq i \leq s)$ is an irreducible matrix and $n_1 + \cdots + n_s = n$. Consequently, if a vertex u corresponds to a row of the block X_{n_i}, then the same block contains the row related to each vertex v being at the largest distance from u. By $\mathrm{sp}(X_{n_i}) \subseteq \mathrm{sp}_{\mathcal{E}}(G)$ we deduce that ξ_1 and ξ_2 both belong to $\mathrm{sp}(X_{n_i})$ for each $1 \leq i \leq s$. The Perron-Frobenius Theorem implies that the multiplicity of ξ_1 and ξ_2 for X_{n_i} are 1 and $n_i - 1$ respectively; moreover, there exists an unit all-positive column vector in $\mathbb{R}^{n_i}$ such that $X_{n_i}\alpha(i) = \xi_1\alpha(i)$. The matrix $X_{n_i} - \xi_2 I$ is symmetric and has rank 1. Consequently,

$$X_{n_i} = \xi_2 I + (\xi_1 - \xi_2)\,\alpha(i)\,\alpha(i)^{\top}.$$

Since all elements on the principal diagonal of X_{n_i} are null, then $\alpha(i) = \sqrt{c} \cdot \mathbf{1}_{n_i}$, where $c = -\xi_2/(\xi_1 - \xi_2)$, and $\mathbf{1}_{n_i} \in \mathbb{R}^{n_i}$ is the all-ones column vector. Thereby,

$$X_{n_i} = -\xi_2 (J - I)_{n_i},$$

leading to $\xi_1 = -\xi_2(n_i - 1)$ for each $i = 1, \ldots, s$. In other words, $n_1 = \cdots = n_s := r \geq 2$; $d = -\xi_2$; $\xi_1 = -d(r-1)$ and $\mathcal{E}(G) = dI_{\frac{n}{r}} \otimes (J_r - I_r)$ as claimed. $\square$

From Lemmas 3.4 and 3.5 we easily deduce the announced $\mathcal{E}$-spectral characterization of r-antipodal graphs.

Theorem 3.6. *Let (d, r) be a pair of positive integers with $r \geq 2$. A connected graph G of order $n = rh$ is r-antipodal with diameter d if and only if G has two distinct $\mathcal{E}$-eigenvalues. If this is the case, the $\mathcal{E}$-spectrum of G is given by*

$$\mathrm{sp}_{\mathcal{E}}(G) = \begin{Bmatrix} d(r-1) & -d \\ h & h(r-1) \end{Bmatrix}. \tag{7}$$

The so-called Capobianco's problem was formulated in 1980 (see [14]). It consists in seeking out a characterization for SC-graphs more useful than the equality between radius and diameter. Theorem 3.6 goes precisely in that direction. It is worthwhile to notice that the $\mathcal{E}$-spectrum (7) of an r-antipodal graph allows to uniquely determine its diameter, which is an $\mathcal{E}$-cospectral invariant (see Lemma 4.2). Later on, such a fact will be beneficial for the $\mathcal{E}$-spectral determination of r-antipodal graphs.

3.3. Further discussion on r-antipodal graphs

We now address another question on r-antipodal graphs.

Problem 2. Do r-antipodal graphs admit other characterizations besides Theorem 3.6?

Given any r-antipodal graph G, we say that the number r is the *block index* of G, and we shall sometimes denote it by $\gamma(G)$. This number is determined by the A-eigenvalues of graphs, as the next proposition shows.

Proposition 3.7. [21, Theorem 3.5] *Let G be a connected regular graph of order n and diameter d with distinct A-eigenvalues $\lambda_0 > \lambda_1 > \cdots > \lambda_k$. Then G is an r-antipodal distance-regular graph if and only if $k = d$, and the distance-d graph $\mathcal{G}_d$ is constituted by disjoint copies of the complete graph K_r with*

$$r = 2n\left(\sum_{i=0}^{d}\frac{\pi_0}{\pi_i}\right)^{-1},$$

where π_i $(0 \le i \le d)$ is defined in (4).

Note that for both r-antipodal graphs and distance regular graphs with diameter d, the cardinality of

$$G_d(v) := \{u \in V(G) \mid d(u,v) = d\}$$

does not depend on v. We denote it by δ_d. If G is a distance-regular graph with intersection array $\{b_0\, b_1 \cdots b_{d-1}; c_1\, c_2 \cdots c_d\}$, then

$$\delta_d^* := \delta_d = \frac{b_0\, b_1 \cdots b_{d-1}}{c_1\, c_2 \cdots c_d}.$$

If G is r-antipodal, then $r = \gamma(G) = \delta_d + 1$; hence, by Proposition 3.7,

$$\delta_d = 2n\left(\sum_{i=0}^{d}\frac{\pi_0}{\pi_i}\right)^{-1} - 1.$$

The next proposition recaps many properties we got so far on antipodal and distance-regular graphs.

Proposition 3.8. *Let G be a connected graph with order n.*

(i) *If G is r-antipodal, then the block index $\gamma(G) = r$ divides n and $\gamma(G) = \delta_d + 1$.*
(ii) *If G is a distance-regular graph, then $\gamma(G) = \delta_d^* + 1$.*
(iii) *A distance-regular graph has two distinct $\mathcal{E}$-eigenvalues if and only if there exists a vertex labeling such that $\mathcal{E}(G) = dI_{n/(\delta_d+1)} \otimes (J_{\delta_d+1} - I_{\delta_d+1})$ with $(\delta_d + 1)|n$.*
(iv) *A distance-regular graph G with two distinct $\mathcal{E}$-eigenvalues is r-antipodal. Particularly, G is 2-antipodal if and only if $\delta_d = 1$ if and only if $n = \sum_{i=0}^{d} \frac{\pi_0}{\pi_i}$.*

We now give some examples of r-antipodal graphs, starting from some somehow extremal cases. As already observed, if a graph G is n-antipodal, then necessarily $G = K_n$. Let now $r < n$.

Example 3.3.1. $d \geq 2$ and $r = 2$.

By definition, 2-antipodal graphs are precisely the even graphs. Some examples are: the cycle C_{2t}, the hypercube Q_t, the cocktail-party graph $CP(t)$, the prism $K_2 \times C_{2t}$, the anti-prism A_{4t+2} [25] and the graphs mentioned in Section 8. By means of the Cartesian product, we can construct many other examples. In fact, a graph $G \times H$ is 2-antipodal if and only if both G and H are 2-antipodal [25]. For instance, all graphs of type $C_{2t} \times Q_t \times CP(t) \times A_{2t+2}$ are 2-antipodal. Another peculiar family of 2-antipodal graphs are the so-called *harmonic 2-antipodal graphs* described in [25].

Example 3.3.2. $d = 2$ and $r \geq 2$.

r-antipodal graphs with diameter 2 are precisely the regular complete k-partite graphs. In order to see this, suppose that G is an r-antipodal with diameter 2. By Lemmas 3.4 and 3.5, its vertex set can be labeled in such a way that $\mathcal{E}(G) = 2I_{n/a} \otimes (J_a - I_a)$ for a suitable $a > 1$ dividing $n = |V(G)|$.

Let u and v be two distinct vertices of G. Since G is connected and $d = \varepsilon(u) = \varepsilon(v) = 2$, if $\epsilon_{uv} = 0$, then $d_{uv} = 1$. Thereby, the distance matrix $\mathcal{D}(G)$ has the following form:

$$\mathcal{D}(G) = \begin{pmatrix} 2(J_a - I_a) & J_a & J_a & \cdots & J_a & J_a \\ J_a & 2(J_a - I_a) & J_a & \cdots & J_a & J_a \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ J_a & J_a & J_a & \cdots & 2(J_a - I_a) & J_a \\ J_a & J_a & J_a & \cdots & J_a & 2(J_a - I_a) \end{pmatrix}.$$

The regular complete k-partite graph $K_{a,a,\ldots,a}$ has such a distance matrix. Since Jin and Zhang have proved in [34] that the complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$ is determined by its $\mathcal{D}$-spectrum, then we conclude that $G = K_{a,a,\ldots,a}$.

Example 3.3.3. Distance-regular graphs.

Table 5
Cubic distance-regular graphs.

Graph	n	Intersection array	block index
Tetrahedral graph	4	$\{3;1\}$	4
Utility graph	6	$\{3,2;1,3\}$	3
Cube	8	$\{3,2,1;1,2,3\}$	2
Petersen graph	10	$\{3,2;1,1\}$	7
Heawood graph	14	$\{3,2,2;1,1,3\}$	5
Pappus graph	18	$\{3,2,2,1;1,1,2,3\}$	3
Desargues graph	20	$\{3,2,2,1,1;1,1,2,2,3\}$	2
Dodecahedral graph	20	$\{3,2,1,1,1;1,1,1,2,3\}$	2
Coxeter graph	28	$\{3,2,2,1;1,1,1,2\}$	7
Tutte-Coxeter graph	30	$\{3,2,2,2;1,1,1,3\}$	9
Foster graph	90	$\{3,2,2,2,2,1,1,1;1,1,1,1,2,2,2,3\}$	3
Biggs-Smith graph	102	$\{3,2,2,2,1,1,1;1,1,1,1,1,1,3\}$	9
Tutte 12-cage	106	$\{3,2,2,2,2,2;1,1,1,1,1,3\}$	33

Some 2-antipodal distance-regular graphs can be found with the aid of Proposition 3.8(iv). We can consider the Taylor graphs (see [7, Section 1.5]), i.e., distance-regular graphs with intersection array $\{k,\mu,1;1,\mu,k\}$; the Hadamard graphs (the explicit construction is in [7, Section 1.8]; the intersection array is $\{2\mu,2\mu-1,\mu,1;1,\mu,2\mu-1,2\mu\}$), and Johnson graphs of type $J(2e,e)$ (see [7, Section 9.1]). The next proposition shows that all 2-antipodal Johnson graphs are of this type.

Proposition 3.9. *The Johnson graph $J(n,e)$ is 2-antipodal if and only if $n=2e$.*

Proof. By [7, Theorem 9.1.2] we know that the Johnson graph $J(n,e)$ has diameter $d=\min\{e,n-e\}$ and intersection array

$$\{(e-1)(n-e-1),(e-2)(n-e-2),\ldots,(e-(d-1))(n-e-(d-1));1^2,2^2,\ldots,d^2\}.$$

Such graph is 2-antipodal if and only if $\delta_d^*=1$. This happens if and only if

$$e-d+1=n-e-d+1=1,$$

from which we deduce that $d=e$ and $n=2e$, as claimed. □

Since the only 2-regular connected graphs are the cycles, the only 2-regular connected graphs which are additionally r-antipodal are C_3 and the even cycles C_{2t} ($t\geq 2$). In the $\mathcal{E}$-spectrum of C_n we only find -1 and 2 (resp. t and $-t$) if $n=3$ (resp. $n=2t$). 3-regular connected graphs (also known as cubical distance-regular graphs) have been detected by Biggs et al. in [5]. We have framed their main result in the next lemma.

Lemma 3.10. [5] *There are just thirteen 3-distance-regular graphs: the tetrahedral graph K_4, the utility graph $K_{3,3}$, the cube, the Petersen graph, the Heawood graph, the Pappus graph, the Desargues graph, the dodecahedral graph, the Coxeter graph, the Tutte-Coxeter graph, the Biggs-Smith graph, the Foster graph and the Tutte 12-cage, as summarized in Table 5.*

The following result follows from Lemma 3.5 and Table 5.

Corollary 3.11. *There are just eight cubic r-antipodal distance-regular graphs: the tetrahedral graph K_4; the utility graph $K_{3,3}$; the cube; the Pappus graph; the Desargues graph; the dodecahedral graph; the Coxeter graph; and the Foster graph.*

4. Constructions of $\mathcal{E}$-cospectral graphs

Solving the spectral determination problem for a class of graphs $\mathcal{G}$ with respect to a given matrix M tantamounts to detecting all graphs which are M-cospectral to a given graph $G \in \mathcal{G}$. The task of identifying M-cospectral graphs is made easier if we have restrictive M-cospectral invariants at our disposal. Our aim in this section is to identify $\mathcal{E}$-cospectral invariants.

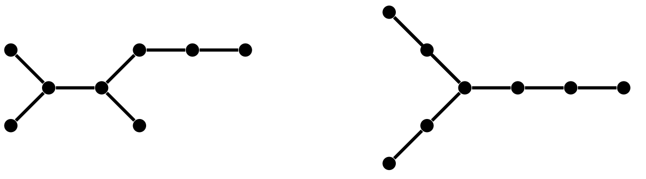

Fig. 3. The smallest pair of $\mathcal{E}$-cospectral trees.

Since the order $|V(G)|$ of a graph G is equal to the cardinality of $\mathrm{sp}_{\mathcal{E}}(G)$, the following lemma is immediate.

Lemma 4.1. *The order of a graph G is an $\mathcal{E}$-cospectral invariant.*

As shown in Fig. 3, the smallest order of $\mathcal{E}$-cospectral trees is 8. By looking at the graphs shown in Fig. 1, we see that a tree with 4 edges is $\mathcal{E}$-cospectral to a non-bipartite graph with 5 edges, and immediately deduce that the number of edges is not an $\mathcal{E}$-cospectral invariant. Furthermore, while it is still unclear whether the $\mathcal{D}$-spectrum determines bipartiteness [36], the same pair of graphs shows that surely this is not the case for the $\mathcal{E}$-spectrum.

We now focus on the $\mathcal{E}$-cospectral invariants related to r-antipodal graphs. There exists just one n-antipodal graph of order n: the complete graph K_n. By Lemmas 3.4 and 3.5, it is straightforward to check that K_n is determined by its $\mathcal{E}$-spectrum. More generally, when $2 \leq r \leq n$, the following lemma holds.

Lemma 4.2. *Let G be a r-antipodal graph of order n. Then the $\mathcal{E}$-cospectral graphs of G are r-antipodal graphs with same order and diameter.*

Proof. Let d be the diameter of G. If a graph H is $\mathcal{E}$-cospectral to G, H has n vertices like G by Lemma 4.1. Moreover $\mathrm{sp}_{\mathcal{E}}(H)$ is given by (7). Theorem 3.6 in particular says that H is r' antipodal, with

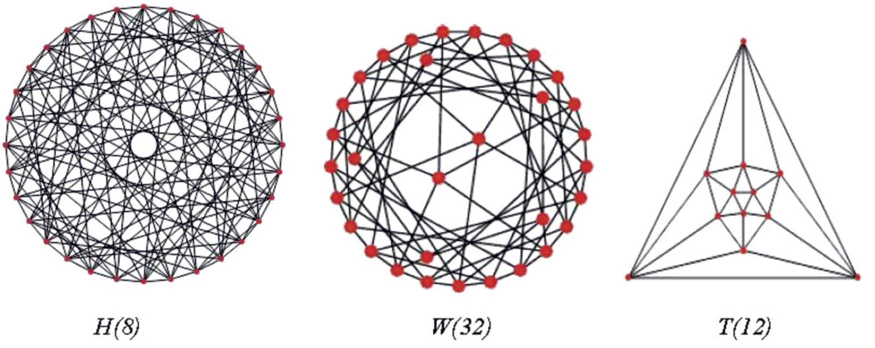

Fig. 4. The Hadamard graph $H(8)$, the Wells graph $W(32)$ and the Taylor graph $T(12)$.

$$(r'-1)\mathrm{diam}(H) = \xi_1(H) = \xi_1(G) = (r-1)d \quad \text{and} \quad -\mathrm{diam}(H) = \xi_2(H) = \xi_2(G) = -d,$$

from which we immediately deduce that $\mathrm{diam}(H) = d$ and $r' = r$. $\square$

The following result easily comes from Theorem 3.6 and Lemma 4.2.

Theorem 4.3. *All r-antipodal graphs ($r \geq 2$) with the same order and diameter are $\mathcal{E}$-cospectral.*

Corollary 4.4.

(i) *In Table 5, the Desargues graph and the dodecahedral graph are $\mathcal{E}$-cospectral graphs. In fact, both graphs have order* 20*, diameter* 5 *and block index* 2*.*
(ii) *The Hadamard graph $H(8)$ with intersection array $\{8, 7, 4, 1; 1, 4, 7, 8\}$ and the Wells graph $W(32)$ with intersection array $\{5, 4, 1, 1; 1, 1, 4, 5\}$ are $\mathcal{E}$-cospectral. In fact, both graphs have order* 32*, diameter* 4 *and block index* 2*.*
(iii) *The Taylor graphs $T(n)$ with intersection array $\{k, \mu, 1; 1, \mu, k\}$ and the complement of the rook's graph $K_2 \times K_{k+1}$ with intersection array $\{k, k-1, 1; 1, k-1, k\}$ are $\mathcal{E}$-cospectral. In fact, both graphs have order $n = 2(k+1)$, diameter* 3 *and block index* 2*.*

The Hadamard graph $H(8)$ and the Wells graph $W(32)$ are depicted in Fig. 4 [47] together with the Taylor graph $T(12)$ with intersection array $\{5, 2, 1; 1, 2, 5\}$. The graph $T(12)$ is $\mathcal{E}$-cospectral to $\overline{K_2 \times K_6}$ with intersection array $\{5, 4, 1; 1, 4, 5\}$.

In order to construct $\mathcal{E}$-cospectral graphs, we can employ products of graphs. Recall that the lexicographic product $G \circ H$ of two graphs G and H is defined as follows: $V(G \circ H) = V(G) \times V(H)$, and two distinct vertices (g, h) and (g', h') of $G \circ H$ are adjacent if and only if either $gg' \in E(G)$ or $g = g'$ and $hh' \in E(H)$. An elementary argument shows $G \circ H$ is connected if and only if G is connected. The next lemma describes how distances of vertices in G, H and $G \circ H$ are related.

Lemma 4.5. [32] *Let G and H be two graphs, and let (g,h) and (g',h') be two vertices of $G \circ H$. Then*

$$d_{G\circ H}((g,h),(g',h')) = \begin{cases} d_G(g,g') & \text{if } g \neq g'; \\ d_H(g,g') & \text{if } g = g' \text{ and } \deg_G(g) = 0; \\ \min\{d_H(h,h'),2\} & \text{if } g = g' \text{ and } \deg_G(g) \neq 0. \end{cases}$$

Lemma 4.6. *Let G be an r-antipodal graph with order $m \geq 2$, $\operatorname{diam}(G) = d$ and $\gamma(G) = r$. Let H be a connected graph with order $n \geq 2$. Then $G \circ H$ is a SC-graph whose diameter is*

$$\operatorname{diam}(G \circ H) = \max\{\, d, \min\{2, \operatorname{diam}(H)\}\,\}. \tag{8}$$

Proof. We can assume that

$$V(G\circ H) = \{(u_i, v_j) | u_i \in V(G), v_j \in V(H), 1 \leq i \leq m, 1 \leq j \leq n\}.$$

In our hypotheses, the degree of any vertex of G and H is at least one, and

$$\min\{d_H(v_j,v_k),2\} = \begin{cases} 1 & \text{if } v_jv_k \in E(H), \\ 2 & \text{if } v_jv_k \notin E(H), \end{cases}$$

since H is connected. This, together with Lemma 4.5, leads to determine the generic element of the distance matrix $\mathcal{D}(G \circ H)$, which is

$$d_{G\circ H}((u_i,v_j),(u_h,v_k)) = \begin{cases} 0 & \text{if } i = h, j = k; \\ 1 & \text{if } i = h, j \neq k \text{ and } v_jv_k \in E(H); \\ 2 & \text{if } i = h, j \neq k \text{ and } v_jv_k \notin E(H); \\ d_G(u_i,u_h) & \text{if } i \neq h. \end{cases} \tag{9}$$

Equation (8) now follows easily. $\square$

It is worth mentioning that Equation (8) or Equation (9) alone suffices to infer that $\operatorname{diam}(G\circ H) = 1$ if and only if G and H are both complete and $|V(G \circ H)| > 1$.

Lemma 4.7. *Let G be an r-antipodal graph with order $m = rm' \geq 2$, $\operatorname{diam}(G) = d \geq 3$ and $\gamma(G) = r$. Let H be a connected graph with order $n \geq 2$. Then there exists a vertex labeling for $V(G \circ H)$ such that $\mathcal{E}(G \circ H) = I_{m'} \otimes ((J_r - I_r) \otimes dJ_n)$. The $\mathcal{E}$-spectrum of $G \circ H$ is*

$$\operatorname{sp}_{\mathcal{E}}(G \circ H) = \begin{Bmatrix} dn(r-1) & 0 & -dn \\ m' & m(n-1) & m'(r-1) \end{Bmatrix}. \tag{10}$$

Proof. Note first that r divides m by definition of r-antipodal graph, and $m' = m/r > 1$, since otherwise G should be complete, but we are supposing $d \geq 3$. Furthermore, for a fixed vertex u_i in G, there exist $r-1$ vertices $u_{c_t} \in V(G)$ $(1 \leq t \leq r-1)$ such that $d_G(u_i, u_{c_t}) = d$. Since $d \geq 3$, by Equation (9) it immediately follows that

$$\epsilon_{G\circ H}((u_i, v_j), (u_h, v_k)) = \begin{cases} 0 & \text{if } i = h, j = k; \\ 0 & \text{if } i = h, j \neq k, v_j v_k \in E(H); \\ 0 & \text{if } i = h, j \neq k, v_j v_k \notin E(H); \\ 0 & \text{if } i \neq h, u_h \notin \{u_{c_1}, \dots, u_{c_{r-1}}\}; \\ d & \text{if } i \neq h, u_h \in \{u_{c_1}, \dots, u_{c_{r-1}}\}. \end{cases} \tag{11}$$

We now consider a vertex labeling for G such that

$$G_d(u_{rh+1}) = \{u_{rh+2}, \dots, u_{rh+r}\}, \qquad \forall\, h = 0, \dots, m'-1.$$

We also order the vertices of $G \circ H$ in a 'lexicographic' way, i.e.

$$(u_i, v_j) \quad \text{precedes} \quad (u_h, v_k) \Longleftrightarrow i < h \quad \text{or } i = h \text{ and } j < k.$$

With this chosen ordering for the elements in $V(G \circ H)$, we have

$$\mathcal{E}(G \circ H) = I_{m'} \otimes ((J_r - I_r) \otimes dJ_n),$$

whose $\mathcal{E}$-eigenvalues are of type $\lambda\mu\nu$, where

$$\lambda \in \operatorname{sp}(I_{m'}) = \begin{Bmatrix} 1 \\ m' \end{Bmatrix}; \quad \mu \in \operatorname{sp}(J_r - I_r) = \begin{Bmatrix} r-1 & -1 \\ 1 & r-1 \end{Bmatrix};$$
$$\nu \in \operatorname{sp}(dJ_n) = \begin{Bmatrix} 0 & dn \\ n-1 & 1 \end{Bmatrix}.$$

Having shown that (10) holds, the proof is over. $\square$

For a graph G, let $[G]_{\mathcal{E}}$ be the set of its $\mathcal{E}$-cospectral mates.

Theorem 4.8. *For $i = 1, 2$, let G_i be an r_i-antipodal graph with diameter at least 3, and let H_1 and H_2 any pair of connected graphs. If $G_2 \in [G_1]_{\mathcal{E}}$ and $H_2 \in [H_1]_{\mathcal{E}}$, then $G_1 \circ H_1$ and $G_2 \circ H_2$ are $\mathcal{E}$-cospectral.*

Proof. By Lemma 4.1 we know that $|G_1| = |G_2| := m$, and $|H_1| = |H_2| := n$. Equalities $r_1 = r_2$ and $\operatorname{diam}(G_1) = \operatorname{diam}(G_2) := d$ come from Lemma 4.2. After suitably labeling the vertex sets of $G_1 \circ H_1$ and $G_2 \circ H_2$, the matrices $\mathcal{E}(G_1 \circ H_1)$ and $\mathcal{E}(G_2 \circ H_2)$ are both equal to

$$I_{\frac{m}{r}} \otimes ((J_r - I_r) \otimes dJ_n)$$

by Lemma 4.7. Consequently, $G_1 \circ H_1$ and $G_2 \circ H_2$ share the same $\mathcal{E}$-spectrum. □

Recall that the *strong product* $G_1 \boxtimes G_2$ of the graphs G_1 and G_2 has vertex set $\{(u_1, u_2)|u_1 \in V(G_1), u_2 \in V(G_2)\}$, and two distinct pairs of vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if one of the following conditions hold: i) $u_1 = v_1$ and u_2 is adjacent to v_2; ii) $u_2 = v_2$ and u_1 is adjacent to v_1; iii) u_1 is adjacent to v_1 and u_2 is adjacent to v_2.

Note that $G_1 \boxtimes G_2 \cong G_2 \boxtimes G_1$ and $G_1 \boxtimes G_2$ is connected if and only if G_1 and G_2 are connected [32].

Lemma 4.9. [44] *Let G be an r-antipodal graph with order m, and $\operatorname{diam}(G) = d \geq 3$. Let H be a graph with order $n \geq 2$. If $d > \operatorname{diam}(H)$, then $G \boxtimes H$ is a SC-graph with diameter d. Moreover, there exists a vertex labeling such that $\mathcal{E}(G \boxtimes H) = I_{\frac{m}{r}} \otimes ((J_r - I_r) \otimes dJ_n)$.*

Theorem 4.10. *Let G_1 and G_2 be $\mathcal{E}$-cospectral r-antipodal graphs with diameter d, and H_1 and H_2 be $\mathcal{E}$-cospectral connected graphs. If $d > \max\{\operatorname{diam}(H_1), \operatorname{diam}(H_2)\}$, then $G_1 \boxtimes H_1$ and $G_2 \boxtimes H_2$ are $\mathcal{E}$-cospectral.*

Proof. Immediately, it follows from Lemmas 4.2 and 4.9. □

The following result follows from Theorems 4.8 and 4.10.

Proposition 4.11. *Let G be an r-antipodal graph and H be a connected graph with $\operatorname{diam}(G) > \operatorname{diam}(H) \geq 2$. Then $G \circ H$ and $G \boxtimes H$ are $\mathcal{E}$-cospectral graphs. Their common $\mathcal{E}$-spectrum is given by (10), where $m' = m/r$.*

Due to Corollary 4.4, Theorems 4.8 and 4.10 and Proposition 4.11, we can construct some new $\mathcal{E}$-cospectral graphs. Some of the graphs evoked in the next corollary are visualized in Fig. 4.

Corollary 4.12. *Let H_1 and H_2 be the Desargues graph and the Dodecahedral graph respectively. Then we can list the following pairs of $\mathcal{E}$-cospectral graphs.*

(i) $H(8) \circ T(12)$ *and* $W(32) \circ \overline{K_2 \times K_6}$, $H_1 \circ T(n)$ *and* $H_2 \circ \overline{K_2 \times K_{k+1}}$.
(ii) $\overline{K_2 \times K_{k+1}} \circ T_1$ *and* $\overline{K_2 \times K_{k+1}} \circ T_2$, $T(n) \circ T_1$ *and* $\overline{K_2 \times K_{k+1}} \circ T_2$.
(iii) $T(n) \boxtimes K_m$ *and* $\overline{K_2 \times K_{k+1}} \boxtimes K_m$, $H(8) \boxtimes T(n)$ *and* $W(32) \boxtimes \overline{K_2 \times K_{k+1}}$.
(vi) $H(8) \circ T(n)$ *and* $H(8) \boxtimes T(n)$; $W(32) \circ \overline{K_2 \times K_{k+1}}$ *and* $W(32) \boxtimes \overline{K_2 \times K_{k+1}}$.

5. Open problems

The first open problem we point out concerns the possible irreducibility of the $\mathcal{E}$-matrix. Proverbially, one of the most basic results of matrix theory applied to graphs

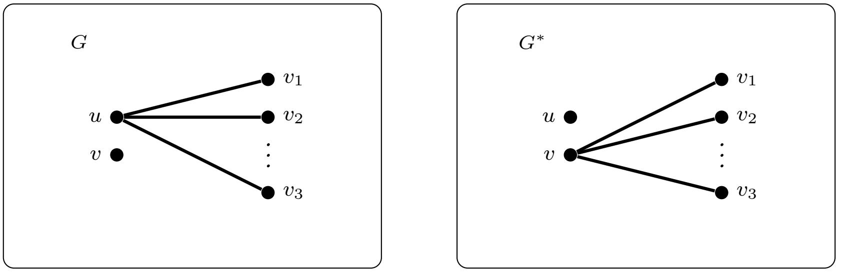

Fig. 5. A graft transformation.

is that the adjacency and distance matrix of connected graphs are both irreducible. Things go differently for the eccentricity matrix. As a matter of fact, $\mathcal{E}(K_n) = J_n - I_n$ is irreducible, nevertheless, $\mathcal{E}(K_{a,b}) = \begin{pmatrix} 2(J_a - I_a) & O \\ O & 2(J_b - I_b) \end{pmatrix}$ is reducible. Thus, the following problem surely deserves some investigation.

Problem 3. For which connected graphs is the eccentricity matrix irreducible?

For the trees, the first two authors among others gave in [45] a complete answer.

Proposition 5.1. [45] *The $\mathcal{E}$-matrix of a tree is irreducible.*

It is well-known that irreducible matrices have a Perron vector, whose components are all positive. The Perron vector is usually applied to inspect whether the spectral radius of graphs decreases or increases under a graft transformation. In order to recall its definition, we denote by $N_G(v)$ the set of the vertices adjacent to v in a connected graph G, and write $N_G[v] = N_G(v) \cup v$. Fix some vertices $v_1, v_2, \dots, v_s$ in $N_G(v) \backslash N_G[u]$, where $\deg(v) \geq 1$ and $u \neq v$; then consider the graph G^* obtained from G by deleting the edges vv_i and adding the edges uv_i $(1 \leq i \leq s)$ (see Fig. 5). The relations between the A-spectral radius and $\mathcal{D}$-spectral radius of G and those of G^* have been studied [43,46]. But a comparison between the $\mathcal{E}$-spectral radii of G and G^* is still to come. The ambitious scholar willing to approach the topic will realize that $\mathcal{E}(G)$ and $\mathcal{E}(G^*)$ may drastically differ.

Problem 4. Investigate the $\mathcal{E}$-spectral radius of graphs under graft transformations.

Problem 5. Given a set $\mathcal{S}$ of graphs, find an upper bound for the $\mathcal{E}$-spectral radius of graphs in $\mathcal{S}$ and characterize the graphs in which the maximal or minimal $\mathcal{E}$-spectral radius is attained.

It is well-known that the coefficients of the A-polynomial of a graph G can be computed through suitable topological invariants of its subgraphs. Such a result was first published in full generality by Sachs ([15, Theorem 1.3]). Thus, it is natural to ask whether a similar formula exists for the coefficients of the $\mathcal{E}$-polynomial of a graph.

Problem 6. Find the relations between the coefficients of the $\mathcal{E}$-polynomial and the subgraphs of a graph.

Focusing our attention to r-antipodal graphs, we have shown that the diameter is determined by the $\mathcal{E}$-spectrum in Section 3. For the general case, the corresponding problem is worth investigating.

Problem 7. Under what conditions is the diameter of a graph determined by $\mathcal{E}$-spectrum?

We finally pose and revisit open problems related to antipodal graphs and SC-graphs. Let G be an even graph of order n and diameter d. Gobel and Veldman [25] showed that $n \geq 4d - 4$ if G is a *symmetric even graph*. Furthermore, they constructed a subfamily of even graphs known in literature as *harmonic even graphs*. For the general case, they proposed the conjecture below, together with the subsequent problem.

Conjecture 3. [25] *Let G be an even graph with order n and diameter d. Then $n \geq 4d-4$.*

Problem 8. [25] For each pair of integers (n, d) with n even, $d \geq 2$ and $n \geq 4d - 4$, is there a procedure to construct all even graphs of order n and diameter d?

As a generalization of the above problem, we think that the following problem deserves consideration as well.

Problem 9. For every pair of integers $n, d \geq 2$, construct all the r-antipodal graphs with order n, diameter d and block index $r|n$.

In this paper, we mainly dealt with SC-graphs, i.e., graphs for which the first inequality in (1) is an equality. A graph G for which the second inequality in (1) is an equality are called *peripheral*. In other words, G is peripheral if and only if $\mathrm{diam}(G) = 2\mathrm{rad}(G)$.

Problem 10. Determine a useful spectral characterization of peripheral graphs.

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