Linear Algebra and its Applications 530 (2017) 485–499

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

On graphs with distance Laplacian spectral radius of multiplicity $n-3$ ☆

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ARTICLE INFO

Article history:
Received 25 March 2017
Accepted 28 May 2017
Available online 1 June 2017
Submitted by D. Stevanovic

MSC:
05C50
05C12
15A18

Keywords:
Distance Laplacian matrix
Laplacian matrix
Largest eigenvalue
Characterised by distance Laplacian spectrum

ABSTRACT

Let $\partial_1^L \geq \partial_2^L \geq \cdots \geq \partial_n^L$ be the distance Laplacian eigenvalues of a connected graph G and $m(\partial_i^L)$ the multiplicity of ∂_i^L. It is well known that the graphs with $m(\partial_1^L) = n-1$ are complete graphs. Recently, the graphs with $m(\partial_1^L) = n-2$ have been characterised by Celso et al. In this paper, we completely determine the graphs with $m(\partial_1^L) = n-3$.

1. Introduction

In this paper we only consider simple connected graphs. Let $G = (V, E)$ be a connected graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{e_1, e_2, \ldots, e_m\}$. The

☆ Supported by the National Natural Science Foundation of China (Grant Nos. 11671344, 11531011).
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http://dx.doi.org/10.1016/j.laa.2017.05.044

distance between v_i and v_j, denoted by $d_G(v_i, v_j)$, is defined as the length of a shortest path between them. The *diameter* of G, denoted by $d(G)$, is the maximum distance between any two vertices of G. The *distance matrix* of G, denoted by $\mathcal{D}(G)$, is the $n \times n$ matrix whose (i, j)-entry is equal to $d_G(v_i, v_j)$, $i, j = 1, 2, \ldots, n$. The *transmission* $Tr(v_i)$ of a vertex v_i is defined as the sum of the distances between v_i and all other vertices in G, that is, $Tr(v_i) = \sum_{j=1}^{n} d_G(v_i, v_j)$. For more details about the distance matrix we refer the readers to [1]. Aouchiche and Hansen [2] introduced the Laplacian for the distance matrix of G as $\mathcal{D}^L(G) = Tr(G) - \mathcal{D}(G)$, where $Tr(G) = diag(Tr(v_1), Tr(v_2), \ldots, Tr(v_n))$ is the diagonal matrix of the vertex transmissions in G. The eigenvalues of $\mathcal{D}^L(G)$, listed by $\partial_1^L \geq \partial_2^L \geq \cdots \geq \partial_n^L = 0$, are called the *distance Laplacian eigenvalues* of G. The multiplicity of ∂_i^L is denoted by $m(\partial_i^L)$. The distance eigenvalues together with their multiplicities is called the *distance Laplacian spectrum* of G, denoted by $Spec_{\mathcal{L}}(G)$.

The distance Laplacian matrix aroused many active studies, such as [1,5,8,9]. Graphs with few distinct eigenvalues form an interesting class of graphs and possess nice combinatorial properties. With respect to distance Laplacian eigenvalues, we focus on the graphs with $m(\partial_1^L)$ being large. Denote by $\mathcal{G}(n)$ the set of connected graphs of order n. Let $\mathcal{G}(n, k) = \{G \in \mathcal{G}(n) \mid m(\partial_1^L) = k\}$ be the set of connected graphs with $m(\partial_1^L) = k$. Aouchiche and Hansen [1] proved that $\mathcal{G}(n, n-1) = \{K_n\}$ and conjectured that $\mathcal{G}(n, n-2) = \{K_{1,n-1}, K_{n/2,n/2}\}$, which has been confirmed by Celso et al. [5]. Motivated by their work, we try to characterise $\mathcal{G}(n, n-3)$. In this paper, we completely determine the graphs in $\mathcal{G}(n, n-3)$ (Theorem 3.3). By the way, we show that all these graphs are determined by their distance Laplacian spectra (Corollary 3.3).

2. Preliminaries

Let G be a connected graph, we always denote by $N_G(v)$ the neighbour set of v in G, that is, $N_G(v) = \{u \in V(G) \mid u \sim v\}$. The i-th largest distance Laplacian eigenvalue of G is denoted by $\partial_i^L(G)$, whose multiplicity is denoted by $m(\partial_i^L(G))$. When it is clear from the context which graph G we mean, we delete G from the notations like $d_G(v_i, v_j)$, $N_G(v)$, $\partial_i^L(G)$ and $m(\partial_i^L(G))$. For a subset $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G induced by S.

As usual, we always write, respectively, K_n, P_n and C_n for the complete graph, the path and the cycle on n vertices. For integers $a_1, a_2, \ldots, a_k \geq 1$, let $K_{a_1,a_2,\ldots,a_k}$ denote the complete k-partite graph on $a_1 + a_2 + \cdots + a_k$ vertices. Let G be a connected graph, denote by $\bar{G}$ the *complement* of G, which is a graph with vertex set $V(\bar{G}) = V(G)$ and two vertices are adjacent whenever they are not adjacent in G. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two connected graphs, the *(disjoint-)union* of G_1 and G_2 is the graph $G_1 \cup G_2$, whose vertex set is $V_1 \cup V_2$ and edge set is $E_1 \cup E_2$. The *join* of G_1 and G_2 is the graph $G_1 \nabla G_2$, which is obtained from $G_1 \cup G_2$ by joining each vertex of G_1 with every vertex of G_2. Moreover, we write $mG = \underbrace{G \cup G \cup \cdots \cup G}_{m}$ for an integer $m \geq 2$.

At first, we introduce the famous Cauchy interlacing theorem.

Theorem 2.1 *([6]). Let A be a real symmetric matrix of order n with eigenvalues $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$ and let M be a principal submatrix of A with order $m \leq n$ and eigenvalues $\lambda_1(M) \geq \lambda_2(M) \geq \cdots \geq \lambda_m(M)$. Then $\lambda_i(A) \geq \lambda_i(M) \geq \lambda_{n-m+i}(A)$, for all $1 \leq i \leq m$.*

Let G be a graph on n vertices, denote by $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \mu_n = 0$ the Laplacian eigenvalues of G and $m(\mu_i)$ the multiplicity of μ_i. There are many pretty properties for Laplacian eigenvalues.

Lemma 2.1 *([3]). Let G be a graph on n vertices with Laplacian eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \mu_n = 0$. Then we have the following results.*
(i) Denote by $m(0)$ the multiplicity of 0 as a Laplacian eigenvalue and $w(G)$ the number of connected components of G. Then $w(G) = m(0)$.
(ii) G has exactly two distinct Laplacian eigenvalues if and only if G is a union of complete graphs of the same order and isolate vertices.
(iii) The Laplacian eigenvalues of $\bar{G}$ are given by $\mu_i(\bar{G}) = n - \mu_{n-i}$ for $i = 1, 2, \ldots, n-1$ and $\mu_n(\bar{G}) = 0$.
(iv) Let H be a graph on m vertices with Laplacian eigenvalues $\mu'_1 \geq \mu'_2 \geq \cdots \geq \mu'_m = 0$, then the Laplacian spectrum of $G\nabla H$ is

$$\{n + m, m + \mu_1, m + \mu_2, \ldots, m + \mu_n, n + \mu'_1, n + \mu'_2, \ldots, n + \mu'_m, 0\}.$$

With respect to distance Laplacian eigenvalues, there are some similar results. The following results are given by Aouchiche and Hansen.

Theorem 2.2 *([2]). Let G be a connected graph on n vertices with $d(G) \leq 2$. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \mu_n = 0$ be the Laplacian spectrum of G. Then the distance Laplacian spectrum of G is $2n - \mu_{n-1} \geq 2n - \mu_{n-2} \geq \cdots \geq 2n - \mu_1 \geq \partial_n^L = 0$. Moreover, for every $i \in \{1, 2, \ldots, n-1\}$ the eigenspaces corresponding to μ_i and to $2n - \mu_i$ are the same.*

Theorem 2.3 *([2]). Let G be a connected graph on n vertices. Then $\partial_{n-1}^L \geq n$ and $\partial_{n-1}^L = n$ if and only if $\bar{G}$ is disconnected. Furthermore, the multiplicity of n as a distance Laplacian eigenvalue is one less than the number of connected components of $\bar{G}$.*

Theorem 2.4 *([2]). Let G be a connected graph on n vertices and $m \geq n$ edges. Consider the connected graph G' obtained from G by the deletion of an edge. Let $\partial_1^L, \partial_2^L, \ldots, \partial_n^L$ and $\partial_1'^L, \partial_2'^L, \ldots, \partial_n'^L$ denote the distance Laplacian spectra of G and G' respectively. Then $\partial_i'^L \geq \partial_i^L$ for all $i = 1, \ldots, n$.*

A graph G is said to be a *cograph* if it contains no induced P_4. There's a pretty result about cographs.

Lemma 2.2 *([4]). Given a graph G, the following statements are equivalent:*
1) G is a cograph.
2) The complement of any connected subgraph of G with at least two vertices is disconnected.
3) Every connected induced subgraph of G has diameter less than or equal to 2.

3. Main results

Recall that $\mathcal{G}(n,k) = \{G \in \mathcal{G}(n) \mid m(\partial_1^L) = k\}$. Aouchiche and Hansen [1] proved that $\mathcal{G}(n,n-1) = \{K_n\}$. Recently, Celso et al. [5] proved that $\mathcal{G}(n,n-2) = \{K_{1,n-1}, K_{n/2,n/2}\}$. They also made efforts to characterise $\mathcal{G}(n,n-3)$. Though they did not give a complete characterisation, their ideas are enlightening. Especially, they proved that the graphs in $\mathcal{G}(n,n-3)$ contain no induced P_5.

Lemma 3.1 *([5], Theorem 4.1). Let $G \in \mathcal{G}(n,n-3)$ with $n \geq 5$ then G does not contain induced P_5.*

Remark 1. If G does not contain induced P_5, then $d(G) \leq 3$. Note that $K_n \notin \mathcal{G}(n,n-3)$. We obtain that $d(G) = 2$ or $d(G) = 3$ for any graph $G \in \mathcal{G}(n,n-3)$ with $n \geq 5$.

Lemma 3.2 *([5], Theorem 3.3). If G is a connected graph then $\partial_1^L \geq \max_{v\in V(G)} Tr(v)+1$ with equality holds if and only if $G \cong K_n$.*

Lemma 3.3. *Let $G \in \mathcal{G}(n,n-3)$ with $n \geq 6$, then ∂_1^L is integral.*

Proof. Let $f(x)$ be the characteristic polynomial of $\mathcal{D}^L(G)$. As $\mathcal{D}^L(G)$ only contains integral entries, we obtain that $f(x)$ is a monic polynomial with integral coefficients. Let $p(x)$ be the minimal polynomial of ∂_1^L, then $p(x) \in Z[x]$ is irreducible in $Q[x]$ and $(p(x))^{n-3}|f(x)$. We assume that $p(x)$ is a polynomial of degree at least 2. Therefore, $p(x)$ has another root $\partial \neq 0$, which is also a distance Laplacian eigenvalue of G with multiplicity $n-3$. It leads to that $n \leq 2(n-3) \leq n-1$, a contradiction. Thus, we have $p(x) = x - \partial_1^L$ and the result follows. □

From Lemmas 3.2 and 3.3, we get the following result.

Corollary 3.1. *Let $G = (V,E) \in \mathcal{G}(n,n-3)$ with $n \geq 6$, then we have $\partial_1^L \geq \max_{v\in V} Tr(v) + 2$. Furthermore, if there exists $v_0 \in V$ such that $\partial_1^L = Tr(v_0) + 2$, then $Tr(v_0) = \max_{v\in V} Tr(v)$.*

Proof. Obviously, $G \neq K_n$. By Lemma 3.2, we have that $\partial_1^L > \max_{v\in V} Tr(v) + 1$. Besides, we get that ∂_1^L is integral from Lemma 3.3. Therefore, we have that $\partial_1^L \geq \max_{v\in V} Tr(v) + 2$. Furthermore, if $\partial_1^L = Tr(v_0) + 2$, then we have $Tr(v_0) + 2 \geq \max_{v\in V} Tr(v) + 2$. It follows that $Tr(v_0) = \max_{v\in V} Tr(v)$. □

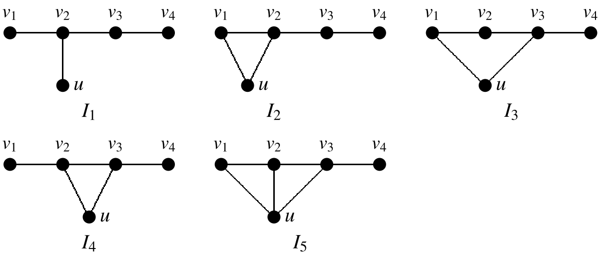

Fig. 1. The graphs $I_1, I_2, \dots, I_5$.

We say that a graph G is P_5*-free* if it does not contain induced P_5. From Lemma 3.1, all graphs in $\mathcal{G}(n, n-3)$ are P_5-free. By Remark 1, a P_5-free graph may have diameter 2 or 3. Now we discuss P_5-free graphs with diameter 3.

Lemma 3.4. *Let G be a connected P_5-free graph on $n \geq 5$ vertices with $d(G) = 3$. Then at least one of I_i for $i = 1, 2, 3, 4, 5$ (shown in Fig. 1) is an induced subgraph of G.*

Proof. Suppose that $d(v_1, v_4) = 3$ and $P = v_1v_2v_3v_4$ is a shortest path from v_1 to v_4. Since $n \geq 5$ and G is connected, there exists $u \in V(G) \backslash V(P)$ such that $N(u) \cap V(P) \neq \emptyset$, where $N(u) = \{v \in V(G) \mid v \sim u\}$ is the neighbour set of u in G. Moreover, since $d(v_1, v_4) = 3$, we have that v_1 and v_4 cannot be adjacent to u simultaneously, that is, $\{v_1, v_4\} \nsubseteq N(u)$. Therefore, we have $1 \leq |N(u) \cap V(P)| \leq 3$.

Assume that $|N(u) \cap V(P)| = 1$. We claim that $N(u) \cap V(P) = \{v_2\}$ or $\{v_3\}$ since G is P_5-free. Both of them lead to the induced subgraph I_1.

Assume that $|N(u) \cap V(P)| = 2$. We claim that $N(u) \cap V(P) = \{v_1, v_2\}$, $\{v_3, v_4\}$, $\{v_1, v_3\}$, $\{v_2, v_4\}$, or $\{v_2, v_3\}$ because $\{v_1, v_4\} \nsubseteq N(u)$. The former two cases lead to the induced subgraph I_2, the next two cases lead to the induced subgraph I_3 and the last case leads to the induced subgraph I_4.

Assume that $|N(u) \cap V(P)| = 3$. We claim that $N(u) \cap V(P) = \{v_1, v_2, v_3\}$ or $\{v_2, v_3, v_4\}$ because $\{v_1, v_4\} \nsubseteq N(u)$. Both cases lead to the induced subgraph I_5. $\square$

Next we introduce a tool which will be used frequently.

Lemma 3.5. *Let $G \in \mathcal{G}(n, n-3)$ with $n \geq 5$ and M a principal submatrix of $\mathcal{D}^L(G)$ of order 5. Then ∂_1^L is also an eigenvalue of M with multiplicity at least two. Furthermore, for each $1 \leq k \leq 5$, there exists an eigenvector $z = (z_1, z_2, \dots, z_5)^T$ of M with respect to ∂_1^L such that $z_k = 0$ and $\sum_{i=1}^{5} z_i = 0$.*

Proof. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_5$ be the eigenvalues of M. By Theorem 2.1, we have $\partial_1^L = \partial_{n-4}^L \leq \lambda_1 \leq \partial_1^L$ and $\partial_1^L = \partial_{n-3}^L \leq \lambda_2 \leq \partial_2^L = \partial_1^L$. Therefore, we have $\lambda_1 = \lambda_2 = \partial_1^L$. Suppose that $x = (x_1, \dots, x_5)^T$ and $y = (y_1, \dots, y_5)^T$ are two independent eigenvectors of M with respect to ∂_1^L. For each fixed integer $1 \leq k \leq 5$, by linear combination of x and y, we get the eigenvector $z = (z_1, \dots, z_5)^T$ satisfying $z_k = 0$. Let

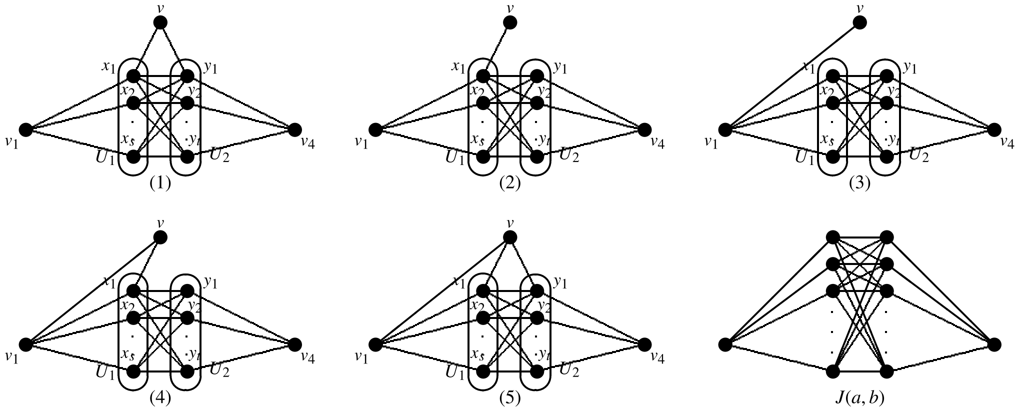

Fig. 2. The graphs used in Lemma 3.6.

$z* = (z_1, \ldots, z_5, 0, \ldots, 0)^T$. Note that $\partial_1^L \geq \frac{z*^T \mathcal{D}^L(G) z*}{z*^T z*} = \frac{z^T M z}{z^T z} = \partial_1^L$. We get that $z*$ is an eigenvector of $\mathcal{D}^L(G)$ with respect to $\partial_1^L \neq 0$. Note that the all-ones vector j is an eigenvector of $\mathcal{D}^L(G)$ with respect to 0. We have $z*^T j = \sum_{i=1}^{5} z_i = 0$. □

Denote by $J(a, b)$ the graph obtained from $K_{1,a} \cup K_{1,b}$ by joining each pendent vertex of $K_{1,a}$ with every pendent vertex of $K_{1,b}$ (shown in Fig. 2). The non-pendent vertices of $K_{1,a}$ and $K_{1,b}$ are called the *roots* of $J(a, b)$.

Lemma 3.6. *Let G be a connected P_5-free graph on $n \geq 5$ vertices with diameter $d(G) = 3$. If none of I_1, I_2, I_4 and I_5 is an induced subgraph of G, then $G = J(a, b)$ for some positive integers $a, b \geq 1$ and $a + b + 2 = n$.*

Proof. Let $d(v_1, v_4) = 3$ and $P = v_1v_2v_3v_4$ a shortest path between v_1 and v_4. By Lemma 3.4, at least one of I_i (shown in Fig. 1) is an induced subgraph of G for $i = 1, 2, \ldots, 5$. Since none of I_1, I_2, I_4 or I_5 is an induced subgraph of G, we obtain that G contains induced I_3.

Note that $I_3 = J(2, 1)$ with roots v_1 and v_4 is an induced subgraph of G. We may assume that $G' = J(a, b)$ with roots v_1 and v_4 is the maximal induced subgraph of G including $J(2, 1)$. Denote by $U_1 = N_{G'}(v_1) = \{x_1, x_2, \ldots, x_a\}$ and $U_2 = N_{G'}(v_4) = \{y_1, y_2, \ldots, y_b\}$. Obviously, $v_2, u \in U_1$ and $v_3 \in U_2$. In what follows we will show that $G = J(a, b)$ with roots v_1 and v_4.

By the way of contradiction, assume that $G \neq J(a, b)$. Then there exists $v \in V(G) \setminus V(G')$ such that $N_G(v) \cap V(G') \neq \emptyset$. Since $d(v_1, v_4) = 3$, v is adjacent to at most one of v_1 and v_4. We claim that v is exactly adjacent to one of v_1 and v_4. Otherwise, we have $v \nsim v_1, v_4$. Then $N_G(v) \cap U_1 \neq \emptyset$ or $N_G(v) \cap U_2 \neq \emptyset$. If v is adjacent to some vertex in $N_G(v) \cap U_1$ and some vertex in $N_G(v) \cap U_2$, say $v \sim x_1$ and $v \sim y_1$ (see Fig. 2 (1)), then we get the induced subgraph $G[v_1, x_1, y_1, v_4, v] = I_4$, a contradiction. If v is only adjacent to some vertex in $N_G(v) \cap U_1$, say $v \sim x_1$ (see Fig. 2 (2)), then we get the induced subgraph $G[v_1, x_1, y_1, v_4, v] = I_1$, a contradiction. If v is only adjacent to some vertex in

$N_G(v) \cap U_2$, say $v \sim y_1$, then we also get the induced subgraph $G[v_1, x_1, y_1, v_4, v] = I_1$, a contradiction. Now we need to consider the following two situations.

Case 1. $v \sim v_1$ and $v \nsim v_4$;

First, we will show that $U_2 \subseteq N_G(v)$. Otherwise, there exists some vertex in U_2 not adjacent to v, say $v \nsim y_1$. Now, if $v \nsim x_1$ (see Fig. 2 (3)), then we get the induced subgraph $G[v, v_1, x_1, y_1, v_4] = P_5$, a contradiction; if $v \sim x_1$ (see Fig. 2 (4)), then we get the induced subgraph $G[v_1, x_1, y_1, v_4, v] = I_2$, a contradiction.

Next we will show that $N_G(v) \cap U_1 = \emptyset$. Otherwise, there exists some vertex in U_1 adjacent to v, say $v \sim x_1$. Recall that $v \sim y_1$ (see Fig. 2 (5)) according to the above arguments, we get the induced subgraph $G[v_1, x_1, y_1, v_4, v] = I_5$, a contradiction.

Summarising the above discussion, we know that $V(G') \cup \{v\}$ induces a subgraph $J(a+1, b)$ of G. This is impossible since $G' = J(a, b)$ is assumed to be the maximal induced subgraph including $J(2, 1)$.

Case 2. $v \sim v_4$ and $v \nsim v_1$;

As similar as Case 1, by symmetry we can also deduce that $G[V(G') \cup \{v\}] = J(a, b+1)$. This is also impossible.

We complete this proof. $\square$

After the completion of the preparations, we get one of our main results.

Theorem 3.1. *Let $G \in \mathcal{G}(n, n-3)$ with $n \geq 6$, then $d(G) = 2$.*

Proof. By Lemma 3.1 and Remark 1, we get that G is P_5-free and $d(G) = 2$ or $d(G) = 3$. Assume by contradiction that $d(G) = 3$. Let $d(v_1, v_4) = 3$ and $P = v_1v_2v_3v_4$ a shortest path between v_1 and v_4. By Lemma 3.4, G contains at least one of I_i (labelled as Fig. 1) as an induced subgraph for $i = 1, 2, \ldots, 5$.

Suppose that I_1 is an induced subgraph of G. Note that $d_G(v_4, u) = d_{I_1}(v_4, u) - 1 = 2$ or $d_G(v_4, u) = d_{I_1}(v_4, u) = 3$. We get that either M_1 or M_1' is a principal submatrix of $\mathcal{D}^L(G)$ with respect to I_1, where

$$M_1 = \begin{pmatrix} t_1 & -1 & -2 & -3 & -2 \\ -1 & t_2 & -1 & -2 & -1 \\ -2 & -1 & t_3 & -1 & -2 \\ -3 & -2 & -1 & t_4 & -2 \\ -2 & -1 & -2 & -2 & t_5 \end{pmatrix} \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ u \end{matrix}, \quad M_1' = \begin{pmatrix} t_1 & -1 & -2 & -3 & -2 \\ -1 & t_2 & -1 & -2 & -1 \\ -2 & -1 & t_3 & -1 & -2 \\ -3 & -2 & -1 & t_4 & -3 \\ -2 & -1 & -2 & -3 & t_5 \end{pmatrix} \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ u \end{matrix}.$$

If M_1 is a principal submatrix of $\mathcal{D}^L(G)$, by Lemma 3.5, there exists an eigenvector $x = (x_1, x_2, x_3, x_4, 0)$ satisfying $x_1 + x_2 + x_3 + x_4 = 0$ such that $M_1 x = \partial_1^L x$. Consider the fifth entry of both sides of $M_1 x = \partial_1^L x$, we have $-2x_1 - x_2 - 2x_3 - 2x_4 = 0$. It follows that $x_2 = 0$ and $x_1 + x_3 + x_4 = 0$. Next we consider the second entry of both sides of $M_1 x = \partial_1^L x$, we have $-x_1 - x_3 - 2x_4 = 0$. It follows that $x_4 = 0$ and $x_1 + x_3 = 0$. We consider the fourth entry of both sides of $M_1 x = \partial_1^L x$, we have $-3x_1 - x_3 = 0$. It follows that $x_1 = x_3 = 0$. Thus, we have $x = 0$, a contradiction. If M_1' is a principal submatrix of $\mathcal{D}^L(G)$, by Lemma 3.5, there exists an eigenvector $y = (0, y_2, y_3, y_4, y_5)^T$ satisfying

$$y_2 + y_3 + y_4 + y_5 = 0 \tag{1}$$

such that

$$M_1' y = \partial_1^L y. \tag{2}$$

Consider the first entry of both sides of Eq. (2), we have

$$-y_2 - 2y_3 - 3y_4 - 2y_5 = 0. \tag{3}$$

Combining (1) and (3), we have $y_2 = y_4$. If $y_2 = y_4 = 0$, we consider the fourth entry of both sides of (2) and we get that $y_5 = 0$. It follows that $y = 0$, a contradiction. If $y_2 = y_4 \neq 0$, we consider the second entry of both sides of (2) and we get that $\partial_1^L = t_2 - \frac{y_3+2y_4+y_5}{y_2} = t_2 - \frac{y_2+y_3+y_4+y_5}{y_2}$. From (1), we have $\partial_1^L = t_2$. It contradicts Corollary 3.1.

Suppose that I_2 is an induced subgraph of G. Note that $d_G(v_4, u) = d_{I_2}(v_4, u) = 3$ or $d_G(v_4, u) = d_{I_2}(v_4, u) - 1 = 2$. We get that the matrix M_2 or M_2' is a principal submatrix of $\mathcal{D}^L(G)$ with respect to I_2, where

$$M_2 = \begin{pmatrix} t_1 & -1 & -2 & -3 & -1 \\ -1 & t_2 & -1 & -2 & -1 \\ -2 & -1 & t_3 & -1 & -2 \\ -3 & -2 & -1 & t_4 & -3 \\ -1 & -1 & -2 & -3 & t_5 \end{pmatrix} \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ u \end{matrix}, \quad M_2' = \begin{pmatrix} t_1 & -1 & -2 & -3 & -1 \\ -1 & t_2 & -1 & -2 & -1 \\ -2 & -1 & t_3 & -1 & -2 \\ -3 & -2 & -1 & t_4 & -2 \\ -1 & -1 & -2 & -2 & t_5 \end{pmatrix} \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ u \end{matrix}.$$

If M_2 is a principal submatrix of $\mathcal{D}^L(G)$, by Lemma 3.5, there exists an eigenvector $x = (x_1, 0, x_3, x_4, x_5)^T$ satisfying $x_1 + x_3 + x_4 + x_5 = 0$ such that $M_2 x = \partial_1^L x$. We successively consider the second, the fourth and the third entries of both sides of $M_2 x = \partial_1^L x$, we get that $x_3 = x_4 = 0$ and $x_1 + x_5 = 0$. If $x_1 = 0$, then $x_5 = 0$ and $x = 0$, a contradiction. If $x_1 \neq 0$, consider the first entry of both sides of $M_2 x = \partial_1^L x$, we get that $\partial_1^L = t_1 - \frac{x_5}{x_1} = t_1 + 1$. It contradicts Corollary 3.1. If M_2' is a principal submatrix of $\mathcal{D}^L(G)$, by Lemma 3.5, there exists an eigenvector $y = (y_1, y_2, y_3, y_4, 0)^T$ satisfying $y_1 + y_2 + y_3 + y_4 = 0$ such that $M_2' y = \partial_1^L y$. Consider the fifth entry of both sides of $M_2' y = \partial_1^L y$, we have $-y_1 - y_2 - 2y_3 - 2y_4 = 0$. It leads to that $y_1 + y_2 = y_3 + y_4 = 0$. If $y_3 = y_4 = 0$, we consider the third entry of both sides of $M_2' y = \partial_1^L y$ and we get that $y_1 = y_2 = 0$. It leads to that $y = 0$, a contradiction. If $y_1 = y_2 = 0$, we consider the second entry of both sides of $M_2' y = \partial_1^L y$ and we get that $y_3 = y_4 = 0$. It leads to that $y = 0$, a contradiction. If $y_1, y_2, y_3, y_4 \neq 0$, without loss of generality, we may suppose that $y = (a, -a, 1, -1, 0)$. Consider the third entry of both sides of $M_2' y = \partial_1^L y$, we have

$\partial_1^L = t_3 + 1 - a$. By Corollary 3.1, we have $a < 0$. Consider the fourth entry of both sides of $M_2' y = \partial_1^L y$, we have $\partial_1^L = t_4 + 1 + a$. By Corollary 3.1, we have $a > 0$, a contradiction.

Suppose that I_4 is an induced subgraph of G. We get that the matrix M_4 is a principal submatrix of $\mathcal{D}^L(G)$ with respect to I_4, where

$$M_4 = \begin{pmatrix} t_1 & -1 & -2 & -3 & -2 \\ -1 & t_2 & -1 & -2 & -1 \\ -2 & -1 & t_3 & -1 & -1 \\ -3 & -2 & -1 & t_4 & -2 \\ -2 & -1 & -1 & -2 & t_5 \end{pmatrix} \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ u \end{matrix}.$$

By Lemma 3.5, there exists an eigenvector $x = (x_1, 0, x_3, x_4, x_5)^T$ satisfying $x_1 + x_3 + x_4 + x_5 = 0$ such that $M_4 x = \partial_1^L x$. Consider the second and the fourth entries of both sides of $M_4 x = \partial_1^L x$ successively, we get that $x_4 = 0$, $x_1 = x_3$ and $x_5 = -2x_1$. If $x_1 = x_3 = 0$, then $x = 0$, a contradiction. If $x_1 = x_3 \neq 0$, consider the third entry of both sides of $M_4 x = \partial_1^L x$ and we get that $\partial_1^L = t_3$. It contradicts Corollary 3.1.

Suppose that I_5 is an induced subgraph of G. We get that the matrix M_5 is a principal submatrix of $\mathcal{D}^L(G)$ with respect to I_5, where

$$M_5 = \begin{pmatrix} t_1 & -1 & -2 & -3 & -1 \\ -1 & t_2 & -1 & -2 & -1 \\ -2 & -1 & t_3 & -1 & -1 \\ -3 & -2 & -1 & t_4 & -2 \\ -1 & -1 & -1 & -2 & t_5 \end{pmatrix} \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ u \end{matrix}.$$

By Lemma 3.5, there exists an eigenvector $x = (x_1, 0, x_3, x_4, x_5)^T$ satisfying $x_1 + x_3 + x_4 + x_5 = 0$ such that $M_5 x = \partial_1^L x$. We successively consider the second and the fourth entries of both sides of $M_5 x = \partial_1^L x$, we have that $x_4 = 0$, $x_1 = x_3$ and $x_5 = -2x_3$. If $x_3 = 0$, we have $x = 0$, a contradiction. If $x_3 \neq 0$, consider the third entry of both sides of $M_5 x = \partial_1^L x$, we have $\partial_1^L = t_3$. It contradicts Corollary 3.1.

Suppose that I_3 is an induced subgraph of G. On the one hand, we get that the matrix M_3 is a principal submatrix of $\mathcal{D}^L(G)$ with respect to I_3, where

$$M_3 = \begin{pmatrix} t_1 & -1 & -2 & -3 & -1 \\ -1 & t_2 & -1 & -2 & -2 \\ -2 & -1 & t_3 & -1 & -1 \\ -3 & -2 & -1 & t_4 & -2 \\ -1 & -2 & -1 & -2 & t_5 \end{pmatrix} \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ u \end{matrix}.$$

By Lemma 3.5, there exists an eigenvector $x = (x_1, x_2, 0, x_4, x_5)^T$ satisfying $x_1 + x_2 + x_4 + x_5 = 0$ such that $M_3 x = \partial_1^L x$. We successively consider the third, the first and the fourth entries of both sides of $M_3 x = \partial_1^L x$, then we get that $x_1 = x_4 = 0$ and $x_2 + x_5 = 0$.

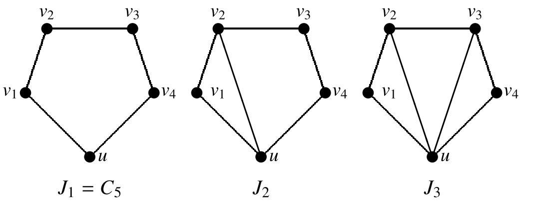

Fig. 3. The graphs in Lemma 3.7.

If $x_2 = x_5 = 0$, then $x = 0$, a contradiction. If $x_2 \neq 0$, without loss of generality, we may suppose that $x = (0, 1, 0, 0, -1)^T$. Consider the second entry of $M_3 x = \partial_1^L x$, we get that $\partial_1^L = t_2 + 2$. By Corollary 3.1, we get that

$$Tr(v_2) = \max_{v \in V(G)} Tr(v). \tag{4}$$

On the other hand, recall that G is P_5-free. Moreover, by the arguments above, we have that G contains no induced I_1, I_2, I_4 or I_5. Therefore, by Lemma 3.6, we have that $G = J(a, b)$ with roots v_1 and v_4. By simple calculation, we have $Tr(v_1) = a + 2b + 3$, $Tr(v_4) = 2a + b + 3$, $Tr(x) = 2a + b + 1$ for every $x \in N(v_1)$ and $Tr(y) = 2b + a + 1$ for every $y \in N(v_4)$. Note that $v_2 \in N(v_1)$. We get that

$$Tr(v_2) = 2a + b + 1 < 2a + b + 3 = Tr(v_4),$$

which contradicts (4).

We complete the proof. $\square$

The result above showed that the graphs in $\mathcal{G}(n, n-3)$ have diameter 2. In fact, we can further obtain that G is the join of two graphs. To prove this, we need the following result.

Lemma 3.7. *Let $G \in \mathcal{G}(n, n-3)$ with $n \geq 6$, then none of $J_1 (= C_5)$, J_2 or J_3 (shown in Fig. 3) can be an induced subgraph of G.*

Proof. Assume by contradiction that $J_1 = C_5$ is an induced subgraph of G. We get that the matrix N_1 is a principal submatrix of $\mathcal{D}^L(G)$ with respect to J_1, where

$$N_1 = \begin{pmatrix} t_1 & -1 & -2 & -2 & -1 \\ -1 & t_2 & -1 & -2 & -2 \\ -2 & -1 & t_3 & -1 & -2 \\ -2 & -2 & -1 & t_4 & -1 \\ -1 & -2 & -2 & -1 & t_5 \end{pmatrix} \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ u \end{matrix}.$$

By Lemma 3.5, there exists an eigenvector $x = (0, x_2, x_3, x_4, x_5)^T$ satisfying $x_2 + x_3 + x_4 + x_5 = 0$ such that $N_1 x = \partial_1^L x$. From the first entry of $N_1 x = \partial_1^L x$, we have $-x_2 -$

$2x_3 - 2x_4 - x_5 = 0$. Therefore, we have $x_3 + x_4 = 0$ and $x_2 + x_5 = 0$. If $x_3 = x_4 = 0$, consider the third entry of both sides of $N_1 x = \partial_1^L x$ and we get that $x_2 = x_5 = 0$. It leads to that $x = 0$, a contradiction. If $x_2 = x_5 = 0$, consider the second entry of both sides of $N_1 x = \partial_1^L x$ and we get that $x_3 = x_4 = 0$. It leads to that $x = 0$, a contradiction. If $x_2, x_3, x_4, x_5 \neq 0$, without loss of generality, we may suppose that $x = (0, a, 1, -1, -a)^T$. Thus, we have

$$\begin{pmatrix} t_1 & -1 & -2 & -2 & -1 \\ -1 & t_2 & -1 & -2 & -2 \\ -2 & -1 & t_3 & -1 & -2 \\ -2 & -2 & -1 & t_4 & -1 \\ -1 & -2 & -2 & -1 & t_5 \end{pmatrix} \begin{pmatrix} 0 \\ a \\ 1 \\ -1 \\ -a \end{pmatrix} = \partial_1^L \begin{pmatrix} 0 \\ a \\ 1 \\ -1 \\ -a \end{pmatrix}. \tag{5}$$

Consider the fourth entry of both sides of (5), we have

$$\partial_1^L = t_4 + a + 1.$$

By Corollary 3.1, we have $a \geq 1$. Consider the fifth entry of both sides of Eq. (5), we have

$$\partial_1^L = t_5 + \frac{1}{a} + 2.$$

By Lemma 3.3, we get that ∂_1^L is integral. Therefore, a and $\frac{1}{a}$ are both integral. Thus, we have $a = 1$ and $\partial_1^L = t_4 + 2 = t_5 + 3$. It follows that

$$t_4 = t_5 + 1. \tag{6}$$

On the other hand, by Lemma 3.5, there also exists an eigenvector $y = (y_1, 0, y_3, y_4, y_5)^T$ satisfying $y_1 + y_3 + y_4 + y_5 = 0$ such that $N_1 y = \partial_1^L y$. From the second entry of $N_1 y = \partial_1^L y$, we have $-y_1 - y_3 - 2y_4 - 2y_5 = 0$. Therefore, we have $y_4 + y_5 = 0$ and $y_1 + y_3 = 0$. If $y_1 = y_3 = 0$ or $y_4 = y_5 = 0$, we also get $y = 0$, a contradiction. If $y_1, y_3, y_4, y_5 \neq 0$, without loss of generality, we may suppose that $y = (b, 0, -b, 1, -1)^T$. Thus, we have

$$\begin{pmatrix} t_1 & -1 & -2 & -2 & -1 \\ -1 & t_2 & -1 & -2 & -2 \\ -2 & -1 & t_3 & -1 & -2 \\ -2 & -2 & -1 & t_4 & -1 \\ -1 & -2 & -2 & -1 & t_5 \end{pmatrix} \begin{pmatrix} b \\ 0 \\ -b \\ 1 \\ -1 \end{pmatrix} = \partial_1^L \begin{pmatrix} b \\ 0 \\ -b \\ 1 \\ -1 \end{pmatrix}. \tag{7}$$

Consider the fourth and the fifth entries of both sides of Eq. (7), we have

$$\partial_1^L = t_4 - b + 1 = t_5 - b + 1.$$

It follows that $t_4 = t_5$, which contradicts (6).

Assume by contradiction that J_2 is an induced subgraph of G. We get that the matrix N_2 is a principal submatrix of $\mathcal{D}^L$ with respect to J_2, where

$$N_2 = \begin{pmatrix} t_1 & -1 & -2 & -2 & -1 \\ -1 & t_2 & -1 & -2 & -1 \\ -2 & -1 & t_3 & -1 & -2 \\ -2 & -2 & -1 & t_4 & -1 \\ -1 & -1 & -2 & -1 & t_5 \end{pmatrix} \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ u \end{matrix}.$$

By Lemma 3.5, there exists an eigenvector $x = (x_1, x_2, x_3, x_4, 0)^T$ satisfying $x_1+x_2+x_3+x_4 = 0$ such that $N_2 x = \partial_1^L x$. We successively consider the fifth, the third, the first and the fourth entries of both sides of $N_2 x = \partial_1^L x$, then we get that $x = 0$, a contradiction.

Assume by contradiction that J_3 is an induced subgraph of G. We get that the matrix N_3 is a principal submatrix of $\mathcal{D}^L$ with respect to J_3, where

$$N_3 = \begin{pmatrix} t_1 & -1 & -2 & -2 & -1 \\ -1 & t_2 & -1 & -2 & -1 \\ -2 & -1 & t_3 & -1 & -1 \\ -2 & -2 & -1 & t_4 & -1 \\ -1 & -1 & -1 & -1 & t_5 \end{pmatrix} \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ u \end{matrix}.$$

By Lemma 3.5, there exists an eigenvector $x = (x_1, x_2, 0, x_4, x_5)^T$ satisfying $x_1 + x_2 + x_4 + x_5 = 0$ such that $N_3 x = \partial_1^L x$. Consider the third, the first, the fourth and the second entries of both sides of $N_3 x = \partial_1^L x$ successively, we get that $x = 0$, a contradiction. □

Using the above tools, we get the following result.

Theorem 3.2. *Let $G \in \mathcal{G}(n, n-3)$ with $n \geq 6$, then $\bar{G}$ is disconnected. It means that G is the join of some connected graphs.*

Proof. By Lemma 2.2, it suffices to show that G contains no induced P_4. Assume by contradiction that G contains an induced $P_4 = v_1v_2v_3v_4$. By Theorem 3.1, we have $d(G) = 2$. Therefore, there exists a vertex $u \in V(G)$ such that $u \sim v_1, v_4$. It follows that at least one of J_1, J_2 and J_3 will be an induced subgraph of G, contradicts Lemma 3.7. □

For any graph $G \in \mathcal{G}(n, n-3)$, we see that G has at most four distinct eigenvalues, and we also have $\partial_{n-1}^L(G) = n$ by Theorems 2.3 and 3.2. Denote by

$$\mathcal{H}_1(n) = \{G \in \mathcal{G}(n, n-3) \mid Spec_{\mathcal{L}}(G) = [(\partial_1^L)^{n-3}, \partial_{n-2}^L, \partial_{n-1}^L = n, \partial_n^L = 0]\},$$

and

$$\mathcal{H}_2(n) = \{G \in \mathcal{G}(n, n-3) \mid Spec_{\mathcal{L}}(G) = [(\partial_1^L)^{n-3}, \partial_{n-2}^L = \partial_{n-1}^L = n, \partial_n^L = 0]\}.$$

Therefore, $\mathcal{H}_1(n)$ and $\mathcal{H}_2(n)$ are the sets of graphs with four and three distinct eigenvalues in $\mathcal{G}(n, n-3)$, respectively. Thus we have the disjoint decomposition

$$\mathcal{G}(n, n-3) = \mathcal{H}_1(n) \cup \mathcal{H}_2(n).$$

Mohammadian [7] gave the following result.

Lemma 3.8 *([7], Theorem 8). Let G be a graph on $n \geq 5$ vertices whose distinct Laplacian eigenvalues are $0 < \alpha < \beta < \gamma$. Then the multiplicity of α is $n-3$ if and only if G is one of the graphs $K_{2,n-2}$, $K_{n/2,n/2} + e$ or $K_{1,n-1} + e$, where $K_{n/2,n/2} + e$ and $K_{1,n-1} + e$ are the graphs obtained from $K_{n/2,n/2}$ and $K_{1,n-1}$, respectively, by adding an edge e joining any two non-adjacent vertices.*

Note that, when $d(G) = 2$, there exists a correspondence between the distance Laplacian spectrum and the Laplacian spectrum of G. We have the following result.

Corollary 3.2. *For an integer $n \geq 6$, we have $\mathcal{H}_1(n) = \{K_{2,n-2}, K_{n/2,n/2}+e, K_{1,n-1}+e\}$, and their distance Laplacian spectra are given by*

$$\begin{cases} \mathrm{Spec}_{\mathcal{L}}(K_{2,n-2}) = \{(2n-2)^{n-3}, n+2, n, 0\} \\ \mathrm{Spec}_{\mathcal{L}}(K_{n/2,n/2} + e) = \{(\frac{3n}{2})^{n-3}, \frac{3n}{2} - 2, n, 0\} \\ \mathrm{Spec}_{\mathcal{L}}(K_{1,n-1} + e) = \{(2n-1)^{n-3}, 2n-3, n, 0\} \end{cases} \tag{8}$$

Proof. Let $G \in \mathcal{H}_1(n)$ and $Spec_{\mathcal{L}}(G) = \{(\partial_1^L)^{n-3}, \partial_{n-2}^L, n, 0\}$ where $\partial_1^L > \partial_{n-2}^L > n$. By Theorem 3.1, we have $d(G) = 2$. Therefore, by Theorem 2.2, the Laplacian spectrum of G is $\{n, 2n-\partial_{n-2}^L, (2n-\partial_1^L)^{n-3}, 0\}$. Thus, we get that $G \in \{K_{2,n-2}, K_{n/2,n/2}+e, K_{1,n-1}+e\}$ from Lemma 3.8. Conversely, note that all of $K_{2,n-2}$, $K_{n/2,n/2} + e$ and $K_{1,n-1} + e$ are the join of two graphs, by Lemma 2.1 (iv) and Theorem 2.2, we obtain their distance Laplacian spectra, which are shown in (8). Therefore, $K_{2,n-2}, K_{n/2,n/2} + e, K_{1,n-1} + e \in \mathcal{H}_1(n)$, and the result follows. □

In what follows we characterise $\mathcal{H}_2(n)$.

Lemma 3.9. *For an integer $n \geq 6$, we have $\mathcal{H}_2(n) = \{K_2\nabla(n-2)K_1, K_1\nabla K_{\frac{n-1}{2},\frac{n-1}{2}}, K_{\frac{n}{3},\frac{n}{3},\frac{n}{3}}\}$, and their distance Laplacian spectra are given by*

$$\begin{cases} \mathrm{Spec}_{\mathcal{L}}(K_2\nabla(n-2)K_1) = \{(2n-2)^{n-3}, n^2, 0\} \\ \mathrm{Spec}_{\mathcal{L}}(K_1\nabla K_{\frac{n-1}{2},\frac{n-1}{2}}) = \{((3n-1)/2)^{n-3}, n^2, 0\} \\ \mathrm{Spec}_{\mathcal{L}}(K_{\frac{n}{3},\frac{n}{3},\frac{n}{3}}) = \{(4n/3)^{n-3}, n^2, 0\} \end{cases} \tag{9}$$

Proof. Let $G \in \mathcal{H}_2(n)$ and $Spec_{\mathcal{L}}(G) = \{(\partial_1^L)^{n-3}, n^2, 0\}$ where $\partial_1^L > n$. By Theorem 3.1, we get that $d(G) = 2$. Therefore, by Theorem 2.2, the Laplacian spectrum of G is $\{n^2, (2n-\partial_1^L)^{n-3}, 0\}$. By Lemma 2.1 (iii), the Laplacian spectrum of $\bar{G}$ is $\{(\partial_1^L - n)^{n-3}, 0^3\}$. By Lemma 2.1 (i), $\bar{G}$ has exactly three components, denoted by G_1, G_2 and G_3. Moreover, by Lemma 2.1 (ii), G_1, G_2 and G_3 are either complete graphs of the same order or isolate vertices. If none of them is an isolate vertex, then $G_1 \cong G_2 \cong G_3 \cong K_{n/3}$. It follows that $G = \overline{3K_{n/3}} = K_{\frac{n}{3},\frac{n}{3},\frac{n}{3}}$. If there is exactly one of them being an isolate vertex, say G_3, then $G_1 \cong G_2 \cong K_{(n-1)/2}$. It follows that $G = \overline{2K_{(n-1)/2} \cup K_1} = K_1 \nabla K_{\frac{n-1}{2},\frac{n-1}{2}}$. If there are exactly two of them being isolate vertices, say G_2 and G_3, then $G_1 \cong K_{n-2}$. It follows that $G = \overline{K_{n-2} \cup 2K_1} = K_2 \nabla (n-2)K_1$. Conversely, note that all of $K_2\nabla(n-2)K_1$, $K_1\nabla K_{\frac{n-1}{2},\frac{n-1}{2}}$ and $K_{\frac{n}{3},\frac{n}{3},\frac{n}{3}}$ are the join of two graphs, by Lemma 2.1 (iv) and Theorem 2.2, we obtain their distance Laplacian spectra, which are shown in (9). Therefore, $K_2\nabla(n-2)K_1, K_1\nabla K_{\frac{n-1}{2},\frac{n-1}{2}}, K_{\frac{n}{3},\frac{n}{3},\frac{n}{3}} \in \mathcal{H}_2(n)$, and the result follows. □

Recall that $\mathcal{G}(n, n-3) = \mathcal{H}_1(n) \cup \mathcal{H}_2(n)$. Combining Corollary 3.2 and Lemma 3.9, we completely determine $\mathcal{G}(n, n-3)$ in the following result.

Theorem 3.3. *For an integer $n \geq 6$, we have*

$$\mathcal{G}(n, n-3) = \{K_{2,n-2}, K_{1,n-1}+e, K_{n/2,n/2}+e, K_2\nabla(n-2)K_1, K_1\nabla K_{\frac{n-1}{2},\frac{n-1}{2}}, K_{\frac{n}{3},\frac{n}{3},\frac{n}{3}}\}.$$

Remark 2. By using the software SageMath, we get the graphs with $m(\partial_1^L) = n-3$ for $n = 4$ and $n = 5$. That is,

$$\begin{cases} \mathcal{G}(4,1) = \{P_4, K_{1,3}+e, K_2\nabla 2K_1\} \\ \mathcal{G}(5,2) = \{K_{2,3}, K_{1,4}+e, K_2\nabla 3K_1, K_1\nabla K_{2,2}, C_5\} \end{cases}.$$

We end up this paper by the following result.

Corollary 3.3. *Let $G \in \mathcal{G}(n, n-3)$ with $n \geq 5$ then G is determined by its distance Laplacian spectrum.*

Proof. Let $H \in \mathcal{G}(n)$ with $Spec_{\mathcal{L}}(H) = Spec_{\mathcal{L}}(G)$. We get that $H \in \mathcal{G}(n, n-3)$. Then, the result follows by pairwise comparing the distance Laplacian spectra of graphs in $\mathcal{G}(n, n-3)$, which are presented in (8) and (9). □

Acknowledgements

The authors are grateful to the referee for helpful comments. This research war supported by the National Natural Science Foundation of China (Grant Nos. 11671344, 11531011).

References

[1] M. Aouchiche, P. Hansen, Distance spectra of graphs: a survey, Linear Algebra Appl. 458 (2014) 301–384.
[2] M. Aouchiche, P. Hansen, Two Laplacians for the distance matrix of a graph, Linear Algebra Appl. 439 (2013) 21–33.
[3] D. Cvetković, P. Rowlinson, S. Simić, An Introduction to the Theory of Graph Spectra, Cambridge Univ. Press, New York, 2010.
[4] D. Corneil, H. Lerchs, L. Burlingham, Complement reducible graphs, Discrete Appl. Math. 3 (1981) 163–174.
[5] Celso M. Da Silva Jr., et al., A note on a conjecture for the distance Laplacian matrix, Electron. J. Linear Algebra 31 (2016) 60–68.
[6] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge Uni. Press, New York, 1992.
[7] A. Mohammadian, B. Tayfeh-Rezaie, Graphs with four distinct Laplacian eigenvalues, J. Algebraic Combin. 34 (2011) 671–682.
[8] M. Nath, S. Paul, On the distance Laplacian spectra of graphs, Linear Algebra Appl. 460 (2014) 97–110.
[9] F. Tian, D. Wong, J. Rou, Proof for four conjectures about the distance Laplacian and distance signless Laplacian eigenvalues of a graph, Linear Algebra Appl. 471 (2015) 10–20.