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On the Existence of Non-Complete L-Borderenergetic Graphs*

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Abstract

The energy of a graph is defined as the sum of the absolute values of the eigenvalues of its adjacency matrix. A graph G on n vertices is said to be borderenergetic if its energy equals to the energy of the complete graph K_n. In [12], Tura promote this concept for Laplacian matrices. The Laplacian energy of G, introduced by Gutman and Zhou [5], is given by $LE(G) = \sum_{i=1}^{n} |\mu_i - \bar{d}|$, where μ_i are the Laplacian eigenvalues of G and $\bar{d}$ is the average degree of G. A graph G on n vertices is said to be L-borderenergetic if $LE(G) = LE(K_n)$. In this paper, we first present all non-complete L-borderenergetic graphs of order 4, 5, 6, 7. Then we construct one connected non-complete L-borderenergetic graph on n vertices for each integer $n \geq 4$, which extends the result in [12] and completely confirms the existence of non-complete L-borderenergetic graphs. Particularly, we prove that there are at least $\frac{n}{2} + 4$ non-complete L-borderenergetic graphs of order n for any even integer $n \geq 6$.

1 Introduction

Throughout this paper, all graphs are assumed to be finite, undirected and without loops or multiple edges. Let G be such a graph of order n, and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of its adjacency matrix $A(G)$. Then the *energy* of G, denoted by $E(G)$, is the sum of the absolute values of the eigenvalues of $A(G)$, that is, $E(G) = \sum_{i=1}^{n} |\lambda_i|$. There are many results on energy [3, 7–11] and its applications in several areas, including in chemistry see [6] for more details and the references therein. It is well known that the

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energy of the complete graph K_n is $E(K_n) = 2n - 2$. Recently, Gong, Li, Xu and Gutman [4] introduced the concept of borderenergetic. A graph G on n vertices is said to be *borderenergetic* if its energy equals to that of the complete graph K_n, that is, $E(G) = E(K_n) = 2n - 2$. Moreover, they show that there exist borderenergetic graphs of order n for each integer $n \geq 7$. The *Laplacian energy* of G, induced by Gutman and Zhou [5], is given by $LE(G) = \sum_{i=1}^{n} |\mu_i - \bar{d}|$, where μ_i are the Laplacian eigenvalues of G and $\bar{d}$ is the average degree of G. Similarly, the Laplacian energy of the complete graph K_n is $LE(G) = 2n - 2$ as well. With respect to the Laplacian energy, Tura [12] introduced the concept of L-borderenergetic. A graph G on n vertices is said to be *L-borderenergy* if its Laplacian energy equals to that of the complete graph K_n, that is, $LE(G) = LE(K_n) = 2n - 2$. Moreover, Tura [12] also gives several classes of L-borderenergetic graphs, by the way, he proves that for each $r \geq 1$, there are $2r+1$ graphs, of order $4r + 4$, pairwise L-noncospectral and L-borderenergetic graphs, which confirms the existence of non-complete L-borderenergetic graphs for $n = 4r + 4$. Recently, a kind of threshold graphs were found to be L-borderenergetic in [2]. In this paper, we give the existence of non-complete L-borderenergetic graphs for each integer $n \geq 4$. Furthermore, we prove that there are at least $\frac{n}{2} + 4$ non-complete L-borderenergetic graphs for each even integer $n \geq 6$.

2 L-borderenergetic graphs of order 4, 5, 6 and 7

The only connected graphs on 3 vertices are P_3 and K_3. By simple calculation, we know that $LE(P_3) = 10/3$, which is not equal to $LE(K_3) = 4$. Therefore, there is no non-complete L-borderenergetic graphs on less than 4 vertices. As in [4], by using the computer software SageMath we exhaust all non-complete L-borderenergetic graphs on 4, 5, 6 and 7 vertices in Proposition 1-4, respectively.

Proposition 1. *There are exactly two non-complete L-borderenergetic graphs on* 4 *vertices, which are labelled as* $G_1^{(4)}$ *and* $G_2^{(4)}$ *shown in Fig. 1, where* $LE(G_1^{(4)})$ *and* $LE(G_2^{(4)})$ *equal to* $LE(K_4) = 6$.

Proposition 2. *There is exactly one non-complete L-borderenergetic graph on* 5 *vertices, which is labeled as* $G_1^{(5)}$ *shown in Fig. 2.*

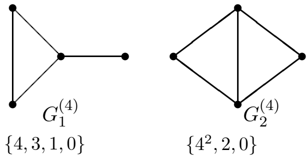


Figure 1. The non-complete L-borderenergetic graphs on 4 vertices and their Laplacian spectra

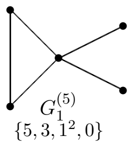


Figure 2. The non-complete L-borderenergetic graph on 5 vertices and its Laplacian spectrum

Proposition 3. *There are exactly* 11 *non-complete L-borderenergetic graphs on* 6 *vertices, which are labeled as* $G_1^{(6)}$, $G_2^{(6)}$,...,$G_{11}^{(6)}$ *depicted in Fig. 3.*

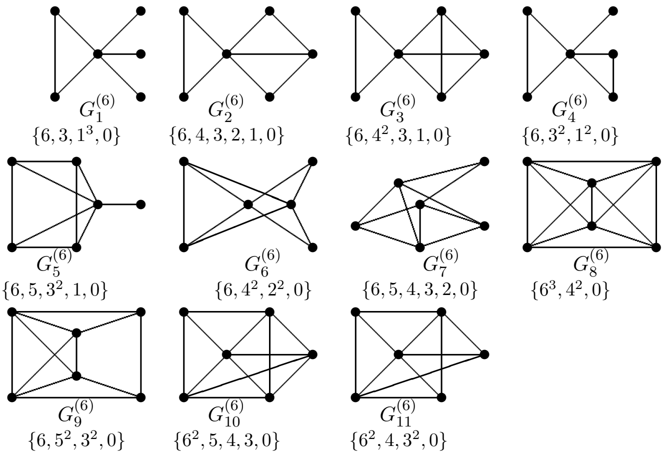
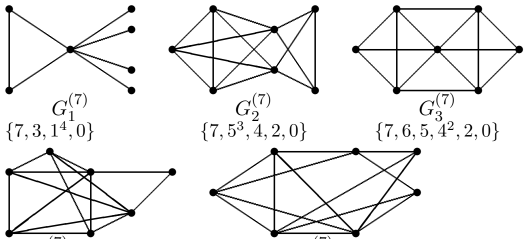


Figure 3. The non-complete L-borderenergetic graphs on 6 vertices and their Laplacian spectra

Proposition 4. *There are exactly* 5 *non-complete L-borderenergetic graphs on* 7 *vertices, which are labeled as* $G_1^{(7)}$, $G_2^{(7)}$,...,$G_5^{(7)}$ *depicted in Fig. 4.*

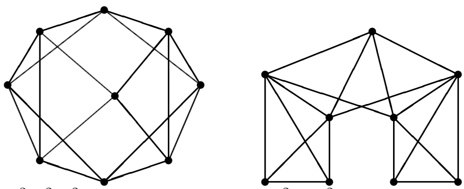


$G_1^{(7)}$ $\{7,3,1^4,0\}$ $G_2^{(7)}$ $\{7,5^3,4,2,0\}$ $G_3^{(7)}$ $\{7,6,5,4^2,2,0\}$

$G_4^{(7)}$ $\{7,6,5,4^2,2,0\}$ $G_5^{(7)}$ $\{7,6,5,4,3^3,0\}$

Figure 4. The non-complete L-borderenergetic graph on 7 vertices and its Laplacian spectrum

Remark 1. *In fact, we also find that there are exactly* 33 *non-complete L-borderenergetic graphs on* 8 *vertices. By simple observation, each L-borderenergetic graph of order less than* 9 *is the join of two graphs. However, there exist non-complete L-borderenergetic graphs which are not the join of two graphs, such as the graphs shown in Fig. 5. By simple calculation, they are L-borderenergetic graphs (their Laplacian spectra and Laplacian energies are given in Fig. 5), but they are not the join of two graphs because their Laplacian spectral radius do not equal to their orders. In the next section, we focus on constructing non-complete L-borderenergetic graphs by using the join operation.*



$\{6^3,5^2,3^2,2,0\}$, $LE = 16$ $\{7,6^2,5,4^2.3,1,0\}$, $LE = 16$

Figure 5. Two non-complete L-borderenergetic graphs which are not the join of two graphs

3 Non-complete L-borderenergetic graphs

In this section, we construct L-borderenergetic graphs by using the graph operations of union and join. We start with the definition of these two operations. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two undirected simple graphs. The *union* $G_1 \cup G_2$ of the graphs G_1 and G_2 is the graph $G = (V, E)$ for which $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$. Denote by mG the union of m's G, that is, $mG = \underbrace{G \cup G \cup \cdots \cup G}_{m}$. The *join* $G_1 \nabla G_2$ of the graphs G_1

and G_2 is obtained from $G_1 \cup G_2$ by joining every vertex in G_1 with every vertex in G_2. The following result is well-known and one can find it in [1].

Lemma 1. *Let G_1 and G_2 be two graphs on n_1 and n_2 vertices, let $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_{n_1} = 0$ and $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_{n_2} = 0$ be the Laplacian eigenvalues of G_1 and G_2, respectively. Then the Laplacian spectra of $G_1 \cup G_2$ and $G_1 \nabla G_2$ are given by*

$$\mathrm{Spec}_L(G_1 \cup G_2) = \{\alpha_1, \ldots, \alpha_{n_1}, \beta_1, \ldots, \beta_{n_2}\},$$
$$\mathrm{Spec}_L(G_1 \nabla G_2) = \{n_1 + n_2, n_2 + \alpha_1, n_2 + \alpha_2, \ldots, n_2 + \alpha_{n_1-1}, n_1 + \beta_1, \ldots, n_1 + \beta_{n_2-1}, 0\}.$$

By a observation of Propositions 1, 2, 3 and 4, many of the L-borderenergetic graphs we depicted are obtained from a graph by joining a new point. Now we construct a graph having such form. For an integer $n \geq 4$, we construct the graph

$$G_n(a,b) = (aK_2 \cup bK_1)\nabla K_1$$

where $a, b \geq 0$ and $2a + b + 1 = n$. Since $n \geq 4$, it is easy to see that $G_n(a,b)$ is a non-complete connected graph. By Lemma 1, we get the Laplacian eigenvalues of $G_n(a,b)$.

Lemma 2. *The Laplacian spectrum of G_n is given by*

$$\mathrm{Spec}_L(G_n(a,b)) = \{n, 3^a, 1^{a+b-1}, 0\}.$$

Proof. It is well known that the Laplacian spectra of K_1 and K_2 are $\{0\}$ and $\{2, 0\}$, respectively. By Lemma 1, we have $Spec_L(aK_2 \cup bK_1) = \{2^a, 0^{a+b}\}$. Therefore, by Lemma 1 again, we have $Spec_L(G_n(a,b)) = Spec_L((aK_2 \cup bK_1)\nabla K_1) = \{n, 3^a, 1^{a+b-1}, 0\}$. ∎

It is easy to see that the average degree of $G_n(a,b)$ is

$$\bar{d} = \frac{4a + b + (2a + b)}{n} = \frac{6a + 2b}{n} = 2 + \frac{2a - 2}{n}. \tag{1}$$

Then we get the Laplacian energy of $G_n(a,b)$.

Theorem 1. *The Laplacian energy of $G_n(a,b)$ is given by*

$$LE(G_n(a,b)) = (2n - 2) + \frac{2(a-1)(b-1)}{n}.$$

Proof. By Lemma 2 and Eq. (1), we have

$$\begin{aligned}
LE(G_n(a,b)) &= \sum_{i=1}^{n} |\mu_i - \bar{d}| = \left(n - 2 - \frac{2a-2}{n}\right) + a\left(1 - \frac{2a-2}{n}\right) \\
&+ (a + b - 1)\left(1 + \frac{2a-2}{n}\right) + \left(2 + \frac{2a-2}{n}\right) \\
&= (2n - 2) + \frac{2(a-1)(b-1)}{n}.
\end{aligned}$$

∎

If $G_n(a,b)$ is L-borderenergetic, from Theorem 1, we have

$$LE(G_n(a,b)) = (2n-2) + \frac{2(a-1)(b-1)}{n} = LE(K_n) = 2n-2,$$

which holds if and only if $a = 1$ or $b = 1$. In fact, we have the following result, which confirms the existence of non-complete L-borderenergetic graphs.

Theorem 2. *Let $\mathcal{G}_n = \{G_n(a,b) \mid a,b \geq 0, 2a+b+1 = n\}$ where $n \geq 4$. If n is odd, then $G_n(1,n-3) = (K_2 \cup (n-3)K_1)\nabla K_1$ is the only L-borderenergetic graph in $\mathcal{G}_n$. If n is even, then $G_n(1,n-3) = (K_2 \cup (n-3)K_1)\nabla K_1$ and $G_n(\frac{n-2}{2},1) = (\frac{n-2}{2}K_2 \cup K_1)\nabla K_1$ are the only L-borderenergetic graphs in $\mathcal{G}_n$.*

Remark 2. *From Theorem 2, we claim that there exists non-complete L-borderenergetic graphs on n vertices for any $n \geq 4$. Besides, $G_n(1,n-3) = G_1^{(n)}$ for $i = 4,5,6,7$, which are depicted in Section 2.*

Note that $G_5(1,2)$ is the only non-complete L-borderenergetic graph on 5 vertices. For an even integer $n \geq 4$ we construct another graph

$$H_n(a,b) = \left(\frac{n}{2} - 1\right) K_1 \nabla (aK_1 \cup (K_1 \nabla bK_1))$$

where $a,b \geq 0$ and $a + b + \frac{n}{2} = n$. Obviously, $H_n(a,b)$ is always a non-complete connected graph since $n \geq 4$. By Lemma 1, we get the Laplacian spectrum of $H_n(a,b)$.

Lemma 3. *The Laplacian spectrum of $H_n(a,b)$ is given by*

$$Spec_L(H_n(a,b)) = \begin{cases} \{n, \frac{n}{2}+b, (\frac{n}{2})^{b-1}, (\frac{n}{2}-1)^{\frac{n}{2}-b}, (\frac{n}{2}+1)^{\frac{n}{2}-2}, 0\}, & b \geq 1 \\ \{n, (\frac{n}{2}+1)^{\frac{n}{2}-2}, (\frac{n}{2}-1)^{\frac{n}{2}}, 0\}, & b = 0 \end{cases}$$

Note that the average degree of $H_n(a,b)$ is given by

$$\bar{d} = \begin{cases} \frac{(\frac{n}{2}-1)(\frac{n}{2}+1)+a(\frac{n}{2}-1)+(\frac{n}{2}-1+b)+b(\frac{n}{2})}{n} = \frac{n}{2} + \frac{2(b-1)}{n}, & b \geq 1 \\ \frac{(\frac{n}{2}-1)(\frac{n}{2}+1)+(\frac{n}{2}+1)(\frac{n}{2}-1)}{n} = \frac{n}{2} - \frac{2}{n}, & b = 0 \end{cases} \tag{2}$$

From Lemma 3 and Eq. (2), we get the Laplacian energy of $H_n(a,b)$.

Theorem 3. *The Laplacian energy of $H_n(a,b)$ is given by*

$$LE(H_n(a,b)) = \begin{cases} 2n-2, & b \geq 1 \\ 2n-2-\frac{4}{n}, & b = 0 \end{cases}$$

Proof. We only consider the case of $b \geq 1$, the other case is very similar. From Lemma 3 and Eq. (2), we have

$$\begin{aligned}(H_n(a,b)) &= \sum_{i=1}^{n} |\mu_i - \bar{d}| = \left(n - \left(\frac{n}{2} + \frac{2(b-1)}{n}\right)\right) \\ &+ \left(b - \frac{2(b-1)}{n}\right) + \left(\frac{n}{2} - 2\right)\left(\left(\frac{n}{2}+1\right) - \left(\frac{n}{2} + \frac{2(b-1)}{n}\right)\right) \\ &+ (b-1)\left(\left(\frac{n}{2} + \frac{2(b-1)}{n}\right) - \frac{n}{2}\right) + \left(\frac{n}{2} - b\right)\left(\left(\frac{n}{2} + \frac{2(b-1)}{n}\right)\right. \\ &- \left.\left(\frac{n}{2} - 1\right)\right) + \left(\frac{n}{2} + \frac{2(b-1)}{n}\right) = 2n - 2\end{aligned}$$

■

Therefore, $H_n(a,b)$ is a non-complete L-borderenergetic graph when $b \geq 1$.

Theorem 4. *Let $\mathcal{H}_n = \{H_n(a,b) \mid a, b \geq 0, a + b = \frac{n}{2}\}$, where $n \geq 4$ is even. Then all graphs but $H_n(\frac{n}{2}, 0)$ in $\mathcal{H}_n$ are non-complete L-borderenergetic graphs.*

Remark 3. *Each pair of graphs in $\mathcal{H}_n$ have different Laplacian spectra, and so they are not isomorphic. For $n = 4$, all non-complete L-borderenergetic graphs are contained in this class, that is, $H_4(1,1) = G_1^{(4)}$ and $H_4(0,2) = G_2^{(4)}$. However, there are many L-borderenergetic graphs which are out of this class for $n \geq 6$.*

In what follows, we will construct some other L-borderenergetic graphs. For an even integer $n \geq 6$, denote by

$$\begin{aligned}J_n(1) &= K_1\nabla(K_1 \cup ((\tfrac{n}{2}-1)K_1\nabla(\tfrac{n}{2}-1)K_1)),\\ J_n(2) &= K_1\nabla((\tfrac{n}{2}-1)K_2 \cup K_1),\\ J_n(3) &= (K_2 \cup (\tfrac{n}{2}-2)K_1)\nabla(K_1\nabla(\tfrac{n}{2}-1)K_1),\\ J_n(4) &= (K_2 \cup (\tfrac{n}{2}-2)K_1)\nabla((K_1\nabla(\tfrac{n}{2}-2)K_1) \cup K_1).\end{aligned}$$

As in the proof of Theorem 2, by calculating the Laplacian energy of $J_n(i)$, we obtain that all of them are L-borderenergetic graphs on n vertices. We present their Laplacian spectra and average degrees in Table 1.

Theorem 5. *For any even integer $n \geq 6$, the graphs $J_n(i)$ for $i = 1, 2, 3, 4$ are connected non-complete L-borderenergetic graphs on n vertices.*

Table 1. L-spectra and average degrees of the constructed L-borderenergetic graphs

Graphs	L-Spectra	Average Degrees
$H_n(a,b)=(\frac{n}{2}-1)K_1\nabla(aK_1\cup(K_1\nabla bK_1))$	$\{n,\frac{n}{2}+b,(\frac{n}{2}+1)^{\frac{n}{2}-2},(\frac{n}{2})^{b-1},(\frac{n}{2}-1)^{\frac{n}{2}-b},0\}$	$\frac{n}{2}+\frac{2(b-1)}{n}$
$J_n(1)=K_1\nabla(K_1\cup((\frac{n}{2}-1)K_1\nabla(\frac{n}{2}-1)K_1))$	$\{n,n-1,(\frac{n}{2})^{n-4},1,0\}$	$\frac{n}{2}$
$J_n(2)=K_1\nabla((\frac{n}{2}-1)K_2\cup K_1)$	$\{n,3^{\frac{n}{2}-1},1^{\frac{n}{2}-1},0\}$	$3-\frac{4}{n}$
$J_n(3)=(K_2\cup(\frac{n}{2}-2)K_1)\nabla(K_1\nabla(\frac{n}{2}-1)K_1)$	$\{n^2,\frac{n}{2}+2,(\frac{n}{2}+1)^{\frac{n}{2}-2},(\frac{n}{2})^{\frac{n}{2}-2},0\}$	$\frac{n}{2}+1$
$J_n(4)=(K_2\cup(\frac{n}{2}-2)K_1)\nabla((K_1\nabla(\frac{n}{2}-2)K_1)\cup K_1)$	$\{n,n-1,\frac{n}{2}+2,(\frac{n}{2}+1)^{\frac{n}{2}-3},(\frac{n}{2})^{\frac{n}{2}-1},0\}$	$\frac{n}{2}+1-\frac{2}{n}$
$G_r=((2rK_1\nabla 2rK_1)\cup K_1)\nabla 2rK_1$	$\{6r+1,6r,(4r+1)^{2r-1},(4r)^{4r-2},2r,0\}$	$4r$

From Table 1, we obtain that each pair of graphs given in Theorems 4 and 5 cannot share the same Laplacian spectrum, which leads to the following result.

Corollary 1. *For an even integer $n \geq 6$, there exists at least $\frac{n}{2}+4$ connected non-complete L-borderenergetic graphs, which are $H_n(a,b)$ for $b = 1, 2, \ldots, \frac{n}{2}$ and $J_n(i)$ for $i = 1, 2, 3, 4$.*

For $n = 6r + 1$, we also find a L-borderenergetic graph different from $(K_2 \cup (n-3)K_1)\nabla K_1$.

Theorem 6. *For any integer $r \geq 1$, the graph $G_r = ((2rK_1 \nabla 2rK_1) \cup K_1)\nabla 2rK_1$ is a connected non-complete L-borderenergetic graph on $6r + 1$ vertices.*

The Laplacian spectrum and average degree of G_r are given in Table 1, and we can immediately calculate the Laplacian energy of G_r, which equals to $12r = 2(6r + 1) - 2$, and the verifying works are omitted here.

References

[1] D. Cvetković, P. Rowlinson, S. Simić, *An Introduction to the Theory of Graph Spectra*, Cambridge Univ. Press, New York, 2010.

[2] B. Deng, X. Li, More on L-borderenergetic graphs, *MATCH Commun. Math. Comput. Chem.* **77** (2017) 115–127.

[3] B. Deng, X. Li, I. Gutman, More on borderenergetic graphs, *Lin. Algebra Appl.* **497** (2016) 199–208.

[4] S. C. Gong, X. Li, G. H. Xu, I. Gutman, B. Furtula, Borderenergetic graphs, *MATCH Commun. Math. Comput. Chem.* **74** (2015) 321–332.

[5] I. Gutman, B. Zhou, Laplacian energy graph of a graph, *Lin. Algebra Appl.* **414** (2006) 29–37.

[6] X. Li, Y. Shi, I. Gutman, *Graph Energy*, Springer, New York, 2012.

[7] X. Li, M. Wei, S. Gong, A computer search for the borderenergetic graphs of order 10, *MATCH Commun. Math. Comput. Chem.* **74** (2015) 333–342.

[8] X. Li, H. Ma, All hypoenergetic graphs with maximum degree at most 3, *Lin. Algebra Appl.* **431** (2009) 2127–2133.

[9] X. Li, H. Ma, All connected graphs with maximum degree at most 3 whose energies are equal to the number of vertices, *MATCH Commun. Math. Comput. Chem.* **64** (2010) 7–24.

[10] X. Li, H. Ma, Hypoenergetic and strongly hypoenergetic k-cyclic graphs, *MATCH Commun. Math. Comput. Chem.* **64** (2010) 41–60.

[11] Z. Shao, F. Deng, Correcting the number of borderenergetic graphs of order 10, *MATCH Commun. Math. Comput. Chem.* **75** (2016) 263–266.

[12] F. Tura, L-borderenergetic graphs, *MATCH Commun. Math. Comput. Chem.* **77** (2017) 37–44.