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On the distance spectra of threshold graphs ☆

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A B S T R A C T

A graph is called a threshold graph if it does not contain induced C_4, P_4 or $2K_2$. Such graphs have numerous applications in computer science and psychology, and they also have nice spectral properties. In this paper, we consider the distance matrix of a connected threshold graph. We show that there are no distance eigenvalues of threshold graphs lying in the interval $(-2, -1)$ and all the eigenvalues, other than -2 or -1, are simple. Besides, we determine all threshold graphs with distinct distance eigenvalues.

1. Introduction

Let Γ be a connected graph with vertex set $V = \{v_1, v_2 \ldots, v_n\}$. The *distance* between v_i and v_j, denoted by $d(v_i, v_j)$ (or $d_{i,j}$ for short), is the length of a shortest path between v_i and v_j. The *neighbourhood* of v_i is the collection of all vertices adjacent to v_i, denoted by $N(v_i)$, that is, $N(v_i) = \{v_j \mid d(v_i, v_j) = 1\}$. The *diameter* of Γ, denoted by $d(\Gamma)$, is the

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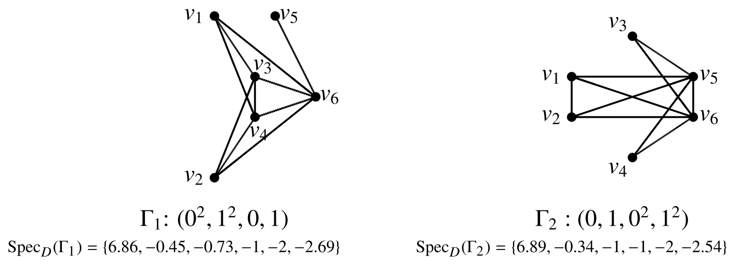

Fig. 1. Example of threshold graphs.

largest distance in Γ. The *distance matrix* of Γ, denoted by $\mathcal{D}(\Gamma)$, is the $n \times n$ matrix whose (i,j)-entry is equal to $d(v_i, v_j)$, for $1 \le i, j \le n$. The eigenvalues of $\mathcal{D}(\Gamma)$, listed by $\partial_1 \ge \partial_2 \ge \cdots \ge \partial_n$, are the *distance eigenvalues* of Γ. The multiset of distance eigenvalues of Γ is the *distance spectrum* of Γ, always denoted by $\mathrm{Spec}_D(\Gamma) = \{[\partial_1]^{m_1}, \ldots, [\partial_s]^{m_s}\}$, where $\partial_1 > \cdots > \partial_s$ and the superscript m_i is the multiplicity of ∂_i. We refer the reader to the survey paper [1], for more details about the backgrounds and applications of distance matrix.

If a graph contains no induced C_4, P_4 or $2K_2$, then it is a *threshold graph*. A threshold graph can be obtained by repeatedly performing one of the following two operations: (a) adding a new vertex adjacent to none of the former vertices (such vertex is called a separate vertex); (b) adding a new vertex adjacent to all of the former vertices (such vertex is called a dominating vertex). Let Γ be a threshold graph of order n, whose vertex set is labelled as $\{v_1, \ldots, v_n\}$ such that v_i is the added vertex in the i-th step of the operations. We can use a $\{0,1\}$-sequence $b = (b_1, \ldots, b_n)$ to represent Γ, where $b_i = 0$ if v_i is a separate vertex and $b_i = 1$ if v_i is a dominating vertex. As usual, we set $b_1 = 0$. Obviously, Γ is connected if and only if $b_n = 1$. Note that the successive separate vertices have the same properties and the successive dominating vertices have the same properties. We collect the successive 0s and 1s together in b. Therefore, the sequence b can be written as $b = (0^{s_1}, 1^{t_1}, \ldots, 0^{s_m}, 1^{t_m})$, where $s_i, t_i \ge 1$ for $1 \le i \le m$ and $m \ge 1$. This sequence is the *representation sequence* of Γ, and Γ is uniquely determined by its representation sequence. For example, the threshold graphs with representation sequences $(0^2, 1^2, 0, 1)$ and $(0, 1, 0^2, 1^2)$ are shown in Fig. 1.

Threshold graphs were first introduced by Chvátal and Hammer [19] and Henderson and Zalcstein [8] in 1977. After that, threshold graphs have been paid close extensive attention because of their numerous applications in computer science and psychology [19]. Recently, many mathematicians studied the eigenvalues of the adjacency matrix of threshold graphs. In 2011, Sciriha and Farrugia [21] gave some spectral properties of adjacency eigenvalues of threshold graphs. In 2013, Bapat [2] obtained the determinant of the adjacency matrix of threshold graphs and he gave the nullity of threshold graphs as well. In the same year, Jacobs et al. [12] presented an $O(n)$ algorithm for constructing a diagonal matrix congruent to $B_x = A + xI$ for any x. By using this method, they published several papers [12,14,15] to investigate the properties of adjacency eigenvalues of threshold graphs from 2013 to 2015. Especially, they obtain the inertia of the adjacency matrix of threshold graphs and they show that all eigenvalues of threshold graph, other than 0 or -1, are simple. In 2017, Banerjee and Mehatari [3] studied the eigenvalues of the normalized matrix (which is similar to the Randić matrix) of threshold graphs.

In this paper, we focus attention on the distance matrix of threshold graphs because the distance matrix seems to contain more informations of a graph than other matrices. It not only reflects whether two vertices are adjacent or not, but also the distance between them. Besides, it is hot topic to study the eigenvalues of the distance matrix of a graph ever since the appearance of the paper [6] by Graham and Pollack, which established a relationship between the number of negative eigenvalues of the distance matrix and the addressing problem in data communication systems. In this paper, we show that all distance eigenvalues of threshold graphs, other than -2 or -1, are simple, a fact reported in [13]. Moreover, we prove that there is no distance eigenvalue that lies in the interval $(-2, -1)$. We obtain formulas for the multiplicities of -2 and -1, the formulas for -2 also reported in [20]. As another main result, we find that there are exactly $\lfloor \frac{n}{2} \rfloor$ threshold graphs with distinct distance eigenvalues and completely determine such graphs as well.

2. Preliminaries

We start with the equitable partition of a symmetric real matrix. The knowledge of an equitable partition contains very rich content and it is a very powerful tool in spectral graph theory. Here we give a brief description of it and we refer the reader to Brouwer and Haemers [4, Section 2.3] or Godsil and Royle [5, Section 9.3] for details.

Suppose that M is an $n \times n$ symmetric real matrix whose rows and columns are indexed by $X = \{1, \ldots, n\}$. Let Π: $X = X_1 \cup \cdots \cup X_m$ be a partition of X. The matrix M can be written as

$$M = \begin{pmatrix} M_{1,1} & \cdots & M_{1,m} \\ \vdots & & \vdots \\ M_{m,1} & \cdots & M_{m,m} \end{pmatrix}$$

where $M_{i,j}$ is the submatrix of M whose rows and columns are induced by X_i and X_j, respectively, for $1 \leq i, j \leq m$. Let b_{ij} be the average row sum of $M_{i,j}$. Then $B_m = (b_{ij})_{m \times m}$ is the *quotient matrix* of M with respect to the partition Π. Especially, if the row sum of each block $M_{i,j}$ is a constant, then the partition is an *equitable partition*. Note that the quotient matrix B_m may not be symmetric even for equitable partitions. The *characteristic matrix* P is the $n \times m$ matrix whose j-th column is the characteristic vector of X_j for $1 \leq j \leq m$. Therefore, it is not hard to verify that $MP = PB_m$. This fact implies the following result.

Lemma 2.1 *([5, Theorem 9.1.1]). Let M be a real symmetric matrix and let Π be an equitable partition of M with quotient matrix B_m. Then the characteristic polynomial of the quotient matrix B_m divides the characteristic polynomial of M.*

Suppose that Γ is a connected graph of order n. Let $\mathcal{D}$ be the distance matrix of Γ and Π: $V = V_1 \cup \cdots \cup V_m$ a partition of the vertex set V. Suppose that $\mathcal{D}_{ij}$ is the submatrix of $\mathcal{D}$ whose rows and columns are induced by V_i and V_j for $1 \leq i, j \leq m$. For $u \in V_i$, the row sum of $\mathcal{D}_{ij}$ corresponding to u is $\sum_{v\in V_j} d(u, v)$. Thus, the partition Π is an equitable partition if, for $1 \leq i, j \leq m$ and $u \in V_i$, the value $\sum_{v\in V_j} d(u, v)$ is a constant independent of the choice of u. In this case, we say that Π is a *distance equitable partition* of Γ. Therefore, we get the following result by applying Lemma 2.1 to distance matrix.

Corollary 2.1. *Let Γ be a connected graph. If Π is a distance equitable partition of Γ with quotient matrix B_m, then all eigenvalues of B_m are distance eigenvalues of Γ.*

Next, we introduce the well-known interlacing theorem, which is another important result in spectral graph theory. A *principal submatrix* of a matrix M is obtained by removing the same corresponding rows and columns from M.

Lemma 2.2 *([5, Theorem 9.3.3]). Let M be a Hermitian matrix of order n, and let H be a principal submatrix of M of order m. If $\theta_1(M) \geq \theta_2(M) \geq \cdots \geq \theta_n(M)$ lists the eigenvalues of M and $\mu_1(H) \geq \mu_2(H) \geq \cdots \geq \mu_m(H)$ the eigenvalues of H, then $\theta_{n-m+i}(M) \leq \mu_i(H) \leq \theta_i(M)$ for $1 \leq i \leq m$.*

At last, we end up this part by the following two results, which are from [18].

Lemma 2.3 *([18, Lemma 3.4]). Let Γ be a connected graph and S a subset of $V(\Gamma)$ with size p. If S is a clique and $N(u) \setminus S = N(v) \setminus S$ for all $u, v \in S$, then -1 is an eigenvalue of $\mathcal{D}(\Gamma)$ with multiplicity at least $p - 1$.*

Lemma 2.4 *([18, Lemma 3.5]). Let Γ be a connected graph and S a subset of $V(\Gamma)$ with size q. If S is an independent set and $N(u) = N(v)$ for all $u, v \in V(S)$, then -2 is an eigenvalue of $\mathcal{D}(\Gamma)$ with multiplicity at least $q - 1$.*

3. The distance eigenvalues of threshold graphs

We may always assume that Γ is a threshold graph with representation sequence $(0^{s_1}, 1^{t_1}, \ldots, 0^{s_m}, 1^{t_m})$, where $s_i, t_j \geq 1$ for $1 \leq i, j \leq m$ and $m \geq 1$. Denote by $s = s_1 + \cdots + s_m$ and $t = t_1 + \cdots + t_m$, so Γ contains $s + t$ vertices. By the expression of the representation sequence of Γ, we have $V(\Gamma) = U_1 \cup V_1 \cup \cdots \cup U_m \cup V_m$, where U_1 is the set of the first s_1 vertices, V_1 is the set of the next t_1 vertices, and so on. In what follows, we always use these notations if there are no additional statements.

By simple observations, we find two special distance eigenvalues of Γ.

Lemma 3.1. *The graph Γ has -2 and -1 as distance eigenvalues with multiplicities at least $s-m$ and $t-m$, respectively.*

Proof. Note that U_i is the set of independent vertices of size s_i for $1 \leq i \leq m$. For $u, v \in U_i$, both of them are adjacent to only the dominating vertices after them. Thus, we have $N(u) = N(v)$. By Lemma 2.4, U_i leads to a distance eigenvalue -2 with multiplicity at least $s_i - 1$. Note that, from the proof of Lemma 2.4 [18], for $i \neq j$, the eigenvectors corresponding to -2 induced by U_i and U_j are pairwise independent. It means that Γ has -2 as a distance eigenvalue with multiplicity at least $(s_1-1)+\cdots+(s_m-1) = s-m$.

Similarly, for $1 \leq j \leq m$, the subset V_i is a clique of order t_j and $N(u)\setminus V_j = N(v)\setminus V_j$ for all $u, v \in V_j$. By Lemma 2.3, V_j leads to a distance eigenvalue -1 with multiplicity at least $t_j - 1$. Note that, from the proof of Lemma 2.3 [18], for $i \neq j$, the eigenvector corresponding to -1 induced by V_i and V_j are pairwise independent. It means that Γ has -1 as a distance eigenvalue with multiplicity at least $(t_1-1)+\cdots+(t_m-1) = t-m$. $\square$

Next, we determine the other distance eigenvalues of Γ by using the equitable partition. Obviously, the graph Γ has a partition Π: $V = U_1 \cup V_1 \cup \cdots \cup U_m \cup V_m$. We show that Π is indeed a distance equitable partition. Note that Γ has diameter 2. By the construction of Γ, for $u \in U_i$ and $x \neq u$, we have

$$d(u,x) = \begin{cases} 2, & x \in U_k, 1 \leq k \leq m \\ 2, & x \in V_k, 1 \leq k \leq i-1 \\ 1, & x \in V_k, i \leq k \leq m \end{cases}.$$

It follows that

$$\sum_{x \in U_j} d(u,x) = \begin{cases} 2(s_i-1), & j = i \\ 2s_j, & j \neq i \end{cases} \text{ and } \sum_{y \in V_j} d(u,y) = \begin{cases} 2t_j, & j \leq i-1 \\ t_j, & j \geq i \end{cases}$$

which are independent of the choice of u. For $v \in V_i$ and $x \neq v$, we have

$$d(v,x) = \begin{cases} 1, & x \in U_k, 1 \leq k \leq i \\ 2, & x \in U_k, i+1 \leq k \leq m \\ 1, & x \in V_k, 1 \leq k \leq m \end{cases}.$$

It follows that

$$\sum_{x \in U_j} d(v,x) = \begin{cases} s_j, & j \leq i \\ 2s_j, & j \geq i+1 \end{cases} \text{ and } \sum_{y \in V_j} d(y,v) = \begin{cases} t_i - 1, & j = i \\ t_j, & j \neq i \end{cases}$$

which are independent of the choice of v. It means that Π is a distance equitable partition with quotient matrix

$$B_m = \begin{pmatrix} 2(s_1-1) & t_1 & 2s_2 & t_2 & \cdots & 2s_m & t_m \\ s_1 & t_1-1 & 2s_2 & t_2 & \cdots & 2s_m & t_m \\ 2s_1 & 2t_1 & 2(s_2-1) & t_2 & \cdots & 2s_m & t_m \\ s_1 & t_1 & s_2 & t_2-1 & \cdots & 2s_m & t_m \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 2s_1 & 2t_1 & 2s_2 & 2t_2 & \cdots & 2(s_m-1) & t_m \\ s_1 & t_1 & s_2 & t_2 & \cdots & s_m & t_m-1 \end{pmatrix}. \tag{1}$$

Note that the columns corresponding to s_i contain only s_i's and columns corresponding to t_i contain only t_i's. By Corollary 2.1, the eigenvalues of B_m are distance eigenvalues of Γ. Thus, we focus on the eigenvalues of B_m. Keep in mind that B_m always denotes the quotient matrix given in (1).

Lemma 3.2. *If $x = (x_1, y_1, \ldots, x_m, y_m)^T$ is an eigenvector of B_m corresponding to eigenvalue λ, then we have*

(i) $(\lambda+1)y_1 = (\lambda+2-s_1)x_1$;

(ii) $(2\lambda+2-t_m)y_m = (\lambda+2)x_m$.

Moreover, if $m \geq 2$, we have

(iii) $t_iy_i = (\lambda+2)(x_{i+1}-x_i)$ *for* $1 \leq i \leq m-1$;

(iv) $s_{i+1}x_{i+1} = (\lambda+1)(y_i - y_{i+1})$ *for* $1 \leq i \leq m-1$.

Proof. From (1) and the equation $B_mx = \lambda x$, we have

$$\begin{cases} \lambda x_1 = & 2(s_1-1)x_1 + t_1y_1 + 2s_2x_2 + t_2y_2 + \cdots + 2s_mx_m + t_my_m \\ \lambda y_1 = & s_1x_1 + (t_1-1)y_1 + 2s_2x_2 + t_2y_2 + \cdots + 2s_mx_m + t_my_m \\ & \cdots\cdots\cdots\cdots\cdots \\ \lambda x_i = & 2s_1x_1 + 2t_1y_1 + \cdots + 2s_{i-1}x_{i-1} + 2t_{i-1}y_{i-1} + 2(s_i-1)x_i + t_iy_i \\ & +2s_{i+1}x_{i+1} + t_{i+1}y_{i+1} + \cdots + 2s_mx_m + t_my_m \\ \lambda y_i = & s_1x_1 + t_1y_1 + \cdots + s_{i-1}x_{i-1} + t_{i-1}y_{i-1} + s_ix_i + (t_i-1)y_i \\ & +2s_{i+1}x_i + t_{i+1}y_i \cdots + 2s_mx_m + t_my_m \\ \lambda x_{i+1} = & 2s_1x_1 + 2t_1y_1 + \cdots + 2s_ix_i + 2t_iy_i + 2(s_{i+1}-1)x_{i+1} + t_{i+1}y_{i+1} \\ & +2s_{i+2}x_{i+2} + t_{i+2}y_{i+2} + \cdots + 2s_mx_m + t_my_m \\ \lambda y_{i+1} = & s_1x_1 + t_1y_1 + \cdots + s_ix_i + t_iy_i + s_{i+1}x_{i+1} + (t_{i+1}-1)y_{i+1} \\ & +2s_{i+2}x_{i+2} + t_{i+2}y_{i+2} \cdots + 2s_mx_m + t_my_m \\ & \cdots\cdots\cdots\cdots\cdots \\ \lambda x_m = & 2s_1x_1 + 2t_1y_1 + 2s_2x_2 + 2t_2y_2 + \cdots + 2(s_m-1)x_m + t_my_m \\ \lambda y_m = & s_1x_1 + t_1y_1 + s_2x_2 + t_2y_2 + \cdots + s_mx_m + (t_m-1)y_m \end{cases}. \tag{2}$$

Subtracting the second equation from the first in (2), we have $\lambda x_1 - \lambda y_1 = (s_1 - 2)x_1 + y_1$, which leads to (i). Multiplying the last equation in (2) by 2 and subtracting from the previous equation, we have $\lambda x_m - 2\lambda y_m = -2x_m - (t_m-2)y_m$, which leads to (ii). Moreover, if $m \geq 2$, for $1 \leq i \leq m-1$, subtracting the (2i-1)-th equation from the (2i+1)-th in (2), we have $\lambda x_{i+1} - \lambda x_i = 2x_i + t_iy_i - 2x_{i+1}$, which leads to (iii). Similarly, we have $\lambda y_{i+1} - \lambda y_i = y_i - s_{i+1}x_i - y_{i+1}$, which leads to (iv). $\square$

By Lemma 3.2, we get some properties of the eigenvalues of B_m.

Corollary 3.1. *If λ is an eigenvalue of B_m, then $\lambda \neq -2$. If, additional, $s_1 \geq 2$, then $\lambda \neq -1$. Furthermore, if $s_1 = 1$, then -1 is a simple eigenvalue of B_m.*

Proof. Let $x = (x_1, y_1, \ldots, x_m, y_m)^T$ be an eigenvector of B_m corresponding to λ.

Suppose to the contrary that $\lambda = -2$. By Lemma 3.2 (ii), since $t_m > 0$, we have $y_m = 0$. If $m = 1$, since $s_1 > 0$, Lemma 3.2 (i) implies $x_1 = 0$ and thus $x = (x_1, y_1)^T = 0$, a contradiction. If $m \geq 2$, since $t_i > 0$, Lemma 3.2 (iii) implies $y_i = 0$ for $1 \leq i \leq m-1$. Therefore, from Lemma 3.2 (i) and (iv), since $s_i > 0$, we have $x_i = 0$ for $1 \leq i \leq m$. It follows that $x = 0$, a contradiction.

Assume that $s_1 \geq 2$. Suppose to the contrary that $\lambda = -1$. If $m = 1$, since $\lambda + 2 - s_1 = 1 - s_1 < 0$, Lemma 3.2 (i) implies $x_1 = 0$. It leads to $y_1 = 0$ by Lemma 3.2 (ii) because $t_1 > 0$. Thus, we have $x = 0$, a contradiction. If $m \geq 2$, since $1 - s_1 < 0$ and $s_i > 0$, Lemma 3.2 (i) and (iv) imply $x_i = 0$ for $1 \leq i \leq m$. This leads to $y_i = 0$ for $1 \leq i \leq m-1$ by Lemma 3.2 (iii) because $t_i > 0$. At last, considering the first equation in (2), we also get $y_m = 0$ because $t_m > 0$. Thus, we have $x = 0$, a contradiction.

Assume that $s_1 = 1$ and $\lambda = -1$. If $m = 1$, then we have $x_1 = -t_1 y_1$ by Lemma 3.2 (ii), and thus -1 is a simple eigenvalue of B_1 with eigenvector $x = (-t_1, 1)^T$. In what follows we assume $m \geq 2$. Since $s_i > 0$, Lemma 3.2 (iv) implies $x_i = 0$ for $2 \leq i \leq m$. It leads to $y_m = 0$ by Lemma 3.2 (ii) because $t_m > 0$. If $m = 2$, then we have $x_1 + t_1 y_1 = 0$ by the last equation in (2). It follows that $x_1 = -t_1 y_1$. Thus, $\lambda = -1$ is a simple eigenvalue of B_2 with eigenvector $x = (-t_1, 1, 0, 0)^T$. If $m \geq 3$, since $x_i = 0$ for $2 \leq i \leq m$ and $t_i > 0$, Lemma 3.2 (iii) implies $y_i = 0$ for $2 \leq i \leq m-1$. By considering the last equation in (2), we have $x_1 + t_1 y_1 = 0$. It follows that $x_1 = -t_1 y_1$ and thus $\lambda = -1$ is a simple eigenvalue of B_m with eigenvector $x = (-t_1, 1, 0, \ldots, 0)^T$. □

Corollary 3.2. *All eigenvalues of B_m are simple.*

Proof. Toward to a contradiction, suppose that λ is an eigenvalue of B_m with multiplicity at least two. Therefore, there exist two independent eigenvectors of B_m corresponding to λ. By taking a linear combination of them, there exists an eigenvector x of B_m corresponding to λ, where $x = (x_1, y_1, \cdots, x_m, y_m)^T$ such that $x_1 = 0$. By Corollary 3.1, we have $\lambda \neq -2$ or -1. Thus, from Lemma 3.2 (i), we have $y_1 = 0$. If $m = 1$, then $x = (x_1, y_1)^T = 0$, a contradiction. If $m \geq 2$, since $x_1 = y_1 = 0$, then $x_2 = 0$ by Lemma 3.2 (iii). Note that Lemma 3.2 (iii) means that $x_{i+1} = 0$ if $x_i = y_i = 0$, and (iv) means that $y_{i+1} = 0$ if $y_i = x_{i+1} = 0$. By using (iii) and (iv) repeatedly, we conclude $x = 0$, a contradiction. □

Corollary 3.3. *There is no eigenvalue of B_m lies in the interval* $(-2, -1)$.

Proof. Suppose to the contrary that $\lambda \in (-2, -1)$ is an eigenvalue of B_m with eigenvector $x = (x_1, y_1, \ldots, x_m, y_m)^T$. Denote by $\alpha = \lambda + 2 \in (0, 1)$. At first, we claim that $x_1 \neq 0$ since otherwise we will get $x = 0$ as the proof of Corollary 3.2. Thus, without loss of generality, suppose that $x_1 = 1$. From Lemma 3.2 (i), we have $y_1 = \frac{\alpha - s_1}{\alpha - 1} > 0$. If $m = 1$, by Lemma 3.2 (ii), we have $y_1 = \frac{\alpha}{2\alpha - 2 - t_1} < 0$, a contradiction. It only needs to consider the case of $m \geq 2$. In what follows, we prove that $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_m)$ are two strictly increase sequences with positive entries. From Lemma 3.2 (iii), we have $x_2 - x_1 = \frac{t_1}{\alpha} y_1 > 0$, and so $x_2 > x_1 > 0$. From Lemma 3.2 (iv), we have $y_1 - y_2 = \frac{s_2}{\alpha - 1} x_2 < 0$, and so $y_2 > y_1 > 0$. Assume that $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_k)$ are two strictly increase sequences with positive entries for some integer k with $k \geq 2$. We consider the sequences $(x_1, \ldots, x_k, x_{k+1})$ and $(y_1, \ldots, y_k, y_{k+1})$. From Lemma 3.2 (iii), we have $x_{k+1} - x_k = \frac{t_k}{\alpha} y_k > 0$, and so $x_{k+1} > x_k > 0$. From Lemma 3.2 (iv), we have $y_k - y_{k+1} = \frac{s_{k+1}}{\alpha - 1} x_{k+1} < 0$, and so $y_{k+1} > y_k > 0$. Thus, we have $x_m, y_m > 0$. However, from Lemma 3.2 (ii) we have $\frac{x_m}{y_m} = \frac{2\alpha - 2 - t_m}{\alpha} < 0$, a contradiction. □

In order to get more properties of the eigenvalues of B_m, an interlacing theorem for B_m is needed. However, the quotient matrix B_m is not symmetric that does not satisfy the condition of the interlacing theorem. The following lemma shows that the interlacing theorem is also true for B_m.

Lemma 3.3. *Let C be a principal submatrix of B_m with size h. If $\lambda_1 \geq \cdots \geq \lambda_{2m}$ lists the eigenvalues of B_m and $\mu_1 \geq \cdots \geq \mu_h$ the eigenvalues of C, then $\lambda_{2m-h+i} \leq \mu_i \leq \lambda_i$ for $1 \leq i \leq h$.*

Proof. Recall that Π: $V = U_1 \cup V_1 \cup \cdots \cup U_m \cup V_m$ is a distance equitable partition of the threshold graph Γ with quotient matrix B_m. Denote by $D = diag(\sqrt{s_1}, \sqrt{t_1}, \ldots, \sqrt{s_m}, \sqrt{t_m})$ the diagonal matrix with the $(2i-1)$-th entry equal to $\sqrt{s_i}$ and the $2i$-th entry equal to $\sqrt{t_i}$ for $1 \leq i \leq m$. Note that the matrix DB_mD^{-1} is obtained by respectively multiplying the $(2i-1)$-th row and the $(2i-1)$-th column of B_m by $\sqrt{s_i}$ and $1/\sqrt{s_i}$, and by respectively multiplying the $2i$-th row and the $2i$-th column of B_m by $\sqrt{t_i}$ and $1/\sqrt{t_i}$. Therefore, we have

$$B'_m = DB_mD^{-1} = \begin{pmatrix} 2(s_1-1) & \sqrt{s_1t_1} & 2\sqrt{s_1s_2} & \cdots & 2\sqrt{s_1s_k} & \sqrt{s_1t_k} \\ \sqrt{t_1s_1} & t_1-1 & 2\sqrt{t_1s_2} & \cdots & 2\sqrt{t_1s_k} & \sqrt{t_1t_k} \\ 2\sqrt{s_2s_1} & 2\sqrt{s_2t_1} & 2(s_2-1) & \cdots & 2\sqrt{s_2s_k} & \sqrt{s_2t_k} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 2\sqrt{s_ks_1} & 2\sqrt{s_kt_1} & 2\sqrt{s_ks_2} & \cdots & 2(s_k-1) & \sqrt{s_kt_k} \\ \sqrt{t_ks_1} & \sqrt{t_kt_1} & \sqrt{t_ks_2} & \cdots & \sqrt{t_ks_k} & t_k-1 \end{pmatrix},$$

which is symmetric. Therefore, B_m and B'_m are similar and have the same eigenvalues, that is, $\lambda_1 \geq \cdots \geq \lambda_{2m}$ are eigenvalues of B'_m. Let C be a principal submatrix of B of size h. Without loss of generality, assume that C is induced by the first h rows and columns of B_m. Therefore, we have $PB_mP^T = C$, where $P = (I_h \mid 0)_{h\times 2m}$ and I_h is the unit matrix. Note that

$$\begin{aligned} C' &= PB'_m P^T = P(DB_m D^{-1})P^T = (PDP^T)PB_m P^T(PD^{-1}P^T) \\ &= (PDP^T)C(PD^{-1}P^T). \end{aligned}$$

Since $(PD^{-1}P^T) = (PDP^T)^{-1}$, we see that C' is similar to C and so $\mu_1 \geq \cdots \geq \mu_h$ are eigenvalues of C'. Note that C' is a principal submatrix of B'_m. By Lemma 2.2, we have $\lambda_{2m-h+i} \leq \mu_i \leq \lambda_i$ for $1 \leq i \leq h$. $\square$

Remark 1. Since adding λI to B shifts the eigenvalues by λ, Lemma 3.3 also implies that the interlacing theorem holds for $B_m + \lambda I$ for any number λ.

Lemma 3.4. *The determinant of $B_m + 2I$ is $(-1)^{m-1}s_1 \cdots s_m t_1 \cdots t_{m-1}(t_m + 2)$.*

Proof. Denote by $row_i(A)$ and $col_j(A)$ the i-th row and the j-th column of a matrix A, respectively. By (1), $B_m + 2I$ is given by

$$B_m + 2I = \begin{pmatrix} 2s_1 & t_1 & 2s_2 & t_2 & \cdots & 2s_m & t_m \\ s_1 & t_1+1 & 2s_2 & t_2 & \cdots & 2s_m & t_m \\ 2s_1 & 2t_1 & 2s_2 & t_2 & \cdots & 2s_m & t_m \\ s_1 & t_1 & s_2 & t_2+1 & \cdots & 2s_m & t_m \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 2s_1 & 2t_1 & 2s_2 & 2t_2 & \cdots & 2s_m & t_m \\ s_1 & t_1 & s_2 & t_2 & \cdots & s_m & t_m+1 \end{pmatrix}.$$

We make some operations on $B_m + 2I$.

In the first step, for $1 \leq i \leq m-1$ we replace row_{2i-1} and row_{2i}, respectively, with $row_{2i-1}(B_m + 2I) - row_{2i+1}(B_m + 2I)$ and $row_{2i}(B_m + 2I) - row_{2i+2}(B_m + 2I)$. Since each subtraction involves rows that differ in at most three places we have:

$$(B_m + 2I)^{(1)} = \begin{pmatrix} 0 & -t_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & s_2 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -t_2 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 2s_1 & 2t_1 & 2s_2 & 2t_2 & \cdots & 2s_m & t_m \\ s_1 & t_1 & s_2 & t_2 & \cdots & s_m & t_m+1 \end{pmatrix}.$$

In the second step, we replace row_{2m-1} with $row_{2m-1}((B_m + 2I)^{(1)}) - 2 \cdot row_{2m}((B_m + 2I)^{(1)})$ obtaining:

$$(B_m+2I)^{(2)} = \begin{pmatrix} 0 & -t_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & s_2 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -t_2 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & -(t_m+2) \\ s_1 & t_1 & s_2 & t_2 & \cdots & s_m & t_m+1 \end{pmatrix}.$$

In the third step, we use the first column to eliminate the entries different from s_1 in the last row. We then use row operations to eliminate the 1's and -1's with $-t_i$, obtaining:

$$(B_m+2I)^{(3)} = \begin{pmatrix} 0 & -t_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & s_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -t_2 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & -(t_m+2) \\ s_1 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Note that these operations do not change the determinant of $B_m + 2I$. We compute $\det((B_m + 2I)^{(3)})$ always expanding along the top row, noting all nonzero terms get multiplied by -1:

$$\begin{aligned} &\det(B_m+2I) = \det((B_m+2I)^{(3)}) \\ &= (-1)^{m-1} t_1 s_2 t_2 s_3 \cdots s_m \det \begin{bmatrix} 0 & -(t_m+2) \\ s_1 & 0 \end{bmatrix} = (-1)^{m-1} s_1 \cdots s_m t_1 \cdots t_{m-1}(t_m+2). \end{aligned}$$

This completes the proof. □

From Lemmas 3.3 and 3.4, we have the following result.

Lemma 3.5. *There are exactly $m-1$ eigenvalues of B_m less than -2.*

Proof. We prove this result by induction on m. If $m = 1$ then the characteristic polynomial of B_1+2I is $\lambda^2-(2s_1+t_1+1)\lambda+s_1t_1+2s_1$. Therefore, we have $\lambda_1\lambda_2 = s_1(t_1+2) > 0$, where $\lambda_1 \geq \lambda_2$ are the eigenvalues of $B_1 + 2I$. Note the eigenvalues of B_1 and therefore $B_1 + 2I$ are real because of Corollary 2.1. It is clear that $\lambda_1 > 0$ and so $\lambda_2 > 0$. It means that there is no eigenvalue of B_1 less than -2 and the result holds. Assume that the result holds for $m = k$ with $k \geq 1$. Suppose that $\mu_1 \geq \cdots \geq \mu_k \geq \mu_{k+1} \geq \mu_{k+2} \geq \cdots \geq \mu_{2k}$ are the eigenvalues of B_k+2I. By assumption, we have $\mu_{k+1} > 0$ and $\mu_{k+2} < 0$. Suppose that $\lambda_1 \geq \cdots \geq \lambda_{k+1} \geq \lambda_{k+2} \geq \lambda_{k+3} \geq \cdots \geq \lambda_{2k+2}$ are eigenvalues of $B_{k+1} + 2I$. Since $B_k + 2I$ is a principal submatrix of $B_{k+1} + 2I$, Lemma 3.2 and Remark 1 imply that

$$\lambda_1 \geq \lambda_{k+1} \geq \mu_{k+1} > 0 \quad \text{and} \quad 0 > \mu_{k+2} \geq \lambda_{k+4} \geq \lambda_{2k+2}.$$

From Lemma 3.4, we have

$$\begin{cases} (-1)^{k-1} s_1 \cdots s_k t_1 \cdots t_{k-1}(t_k+2) = \det(B_k + 2I) = (\mu_1 \cdots \mu_{k+1})(\mu_{k+2} \cdots \mu_{2k}) \\ (-1)^k s_1 \cdots s_{k+1} t_1 \cdots t_k (t_{k+1}+2) = \det(B_{k+1} + 2I) = (\lambda_1 \cdots \lambda_{k+1})(\lambda_{k+2}\lambda_{k+3})(\lambda_{k+4} \cdots \lambda_{2k+2}) \end{cases}.$$

It implies that $\lambda_{k+2}\lambda_{k+3} < 0$, and so $\lambda_{k+2} > 0$ and $\lambda_{k+3} < 0$. Thus, B_{k+1} has exactly k eigenvalues less than -2, which are $\lambda_{k+3} - 2, \ldots, \lambda_{2k+2} - 2$. This completes the proof. □

Combining Corollaries 3.1, 3.2, 3.3 and Lemma 3.5, we obtain our main result.

Theorem 3.1. *Let Γ be a threshold graph with representation sequence $(0^{s_1}, 1^{t_1}, \ldots, 0^{s_m}, 1^{t_m})$ and $s = s_1 + \cdots + s_m$, $t = t_1 + \cdots + t_m$.*

(i) If $s_1 = 1$, then $\mathrm{Spec}_D(\Gamma) = \{\lambda_1, \ldots, \lambda_m, [-1]^{t-m+1}, [-2]^{s-m}, \lambda_{m+2}, \ldots, \lambda_{2m}\}$, where $\lambda_1 > \cdots > \lambda_m > -1 > -2 > \lambda_{m+2} > \cdots > \lambda_{2m}$.

(ii) If $s_1 \geq 2$, then $\mathrm{Spec}_D(\Gamma) = \{\lambda_1, \ldots, \lambda_{m+1}, [-1]^{t-m}, [-2]^{s-m}, \lambda_{m+2}, \ldots, \lambda_{2m}\}$, where $\lambda_1 > \cdots > \lambda_{m+1} > -1 > -2 > \lambda_{m+2} > \cdots > \lambda_{2m}$.

Proof. Let B_m be the quotient matrix of Γ given in (1). By Corollary 3.2 and Lemma 3.5, all eigenvalues of B_m are distinct and there are exactly $m-1$ of them less than -2. Therefore, we assume that $\lambda_1 > \cdots > \lambda_m > \lambda_{m+1} > \lambda_{m+2} > \cdots > \lambda_{2m}$ are the eigenvalues of B_m, where $\lambda_{m+2} < -2$. By Corollary 3.3, we have $\lambda_{m+1} \geq -1$. Moreover, by Corollary 3.1, we have that $\lambda_{m+1} = -1$ if $s_1 = 1$ and $\lambda_{m+1} > -1$ if $s_1 \geq 2$. On the one hand, by Corollary 2.1, the $2m$ eigenvalues of B_m are all eigenvalues of $\mathcal{D}(\Gamma)$. On the other hand, by Lemma 3.1, -2 and -1 are eigenvalues of $\mathcal{D}(\Gamma)$ with multiplicities at least $s-m$ and $t-m$, respectively. Note that $2m + (s-m) + (t-m) = s+t$, which is exactly the order of Γ. We get all the distance eigenvalues of Γ. □

Remark 2. In 2013, Jacobs et al. [13] also stated that all distance eigenvalues other than -1 and -2 are simple. They gave this result according to an algorithm without algebraic proof. Recently, the formula for the multiplicity of -2 was also reported in [20] without proof.

In 1974, Harary and Schwenk [7] proposed an interesting problem: "Which graphs have distinct eigenvalues?" There are only a few results about this problem. Recently, Lou et al. [17] constructed an infinite family of graphs having distinct eigenvalues. With respect to distance eigenvalues, we find all threshold graphs with distinct distance eigenvalues by Theorem 3.1.

Theorem 3.2. *Let Γ be a threshold graph with representation sequence $(0^{s_1}, 1^{t_1}, \ldots, 0^{s_m}, 1^{t_m})$ and $s = s_1 + \cdots + s_m$, $t = t_1 + \cdots + t_m$. Then all distance eigenvalues of Γ are distinct if and only if one of the followings holds.*

Table 1
The threshold graphs with distinct distance eigenvalues on n vertices.

The oder n	$(s_1, t_1, \ldots, s_m, t_m)$	Distance spectrum
5	$(2, 1, 1, 1)$	$\{5.76, -0.56, -0.77, -2, -2.43\}$
5	$(1, 1, 2, 1)$	$\{6.18, -0.64, -1, -2, -2.54\}$
6	$(1, 1, 1, 1, 1, 1)$	$\{7.21, -0.51, -0.76, -1, -2.27, -2.66\}$
6	$(2, 2, 1, 1)$	$\{6.86, -0.45, -0.73, -1, -2, -2.69\}$
6	$(2, 1, 1, 2)$	$\{6.51, -0.35, -0.72, -1, -2, -2.44\}$
7	$(2, 1, 1, 1, 1, 1)$	$\{8.83, -0.34, -0.69, -0.78, -2, -2.28, -2.75\}$
7	$(1, 1, 2, 1, 1, 1)$	$\{9.14, -0.32, -0.75, -1, -2, -2.37, -2.70\}$
7	$(1, 1, 1, 1, 2, 1)$	$\{9.41, -0.51, -0.68, -1, -2, -2.29, -2.92\}$
8	$(1, 1, 1, 1, 1, 1, 1, 1)$	$\{10.31, -0.27, -0.68, -0.78, -1, -2.24, -2.37, -2.98\}$
8	$(2, 2, 1, 1, 1, 1)$	$\{10.06, -0.31, -0.59, -0.76, -1, -2, -2.31, -3.08\}$
8	$(2, 1, 1, 2, 1, 1)$	$\{9.80, -0.12, -0.69, -0.74, -1, -2, -2.37, -2.88\}$
8	$(2, 1, 1, 1, 1, 2)$	$\{9.55, -0.15, -0.60, -0.76, -1, -2, -2.28, -2.77\}$
9	$(2, 1, 1, 1, 1, 1, 1, 1)$	$\{11.92, -0.09, -0.59, -0.73, -0.78, -2, -2.24, -2.40, -3.09\}$
9	$(1, 1, 2, 1, 1, 1, 1, 1)$	$\{12.16, -0.02, -0.64, -0.77, -1, -2, -2.27, -2.48, -2.98\}$
9	$(1, 1, 1, 1, 2, 1, 1, 1)$	$\{12.38, -0.18, -0.62, -0.75, -1, -2, -2.28, -2.42, -3.12\}$
9	$(1, 1, 1, 1, 1, 1, 2, 1)$	$\{12.58, -0.25, -0.64, -0.74, -1, -2, -2.25, -2.43, -3.28\}$

(i) $s_i = t_i = 1$ *for* $i = 1, 2, ..., m$*;*
(ii) $s_1 = 2$*,* $s_i = 1$ *for* $i = 2, 3, ..., m$*, and* $t_j = 1$ *for* $j = 1, 2, ..., m$*;*
(iii) $s_1 = 2$*,* $s_i = 1$ *for* $2 \leq i \leq m$ *and* $t = m + 1$*;*
(iv) $s_1 = 1$*,* $s = m + 1$ *and* $t_j = 1$ *for* $1 \leq j \leq m$*.*

Proof. The sufficiency is immediate By Theorem 3.1. Now we assume that Γ has distinct distance eigenvalues.

Suppose that $-2 \notin \mathrm{Spec}_D(\Gamma)$. By Theorem 3.1, we have $s - m = 0$. It means that $s_i = 1$ for $1 \leq i \leq m$. Therefore, $-1 \in \mathrm{Spec}_D(\Gamma)$ has multiplicity $t - m + 1$. It leads to $t - m = 0$, and so $t_j = 1$ for $1 \leq j \leq m$, (i) holds.

Suppose that $-2 \in \mathrm{Spec}_D(\Gamma)$. Therefore, $s - m = 1$. It means that exactly one of $\{s_1, \ldots, s_m\}$ is 2 and the others are all 1. We first consider the case of $s_1 = 2$. In this case, if $-1 \notin \mathrm{Spec}_D(\Gamma)$, then $t - m = 0$, and so $t_j = 1$ for $1 \leq j \leq m$, (ii) holds; if $-1 \in \mathrm{Spec}_D(\Gamma)$, then $t - m = 1$, and so (iii) holds. Next, we consider the case of $s_1 = 1$. Thus, $-1 \in \mathrm{Spec}_D(\Gamma)$ has multiplicity $t - m + 1$. It implies that $t - m = 0$, and so $t_j = 1$ for $1 \leq j \leq m$, (iv) holds. $\square$

Remark 3. From Theorem 3.2, for a given integer n, there are exactly $\frac{n-1}{2}$ connected threshold graphs with distinct distance eigenvalues for odd n and there are exactly $\frac{n}{2}$ ones for even n. In fact, if n is even, then only (i) and (iii) can be happen. If (i) happen, then $n = 2m$, which leads to only one graph with representation $(0, 1, \ldots, 0, 1)$. If (iii) holds, then $n = 2m + 2$ and so $m = \frac{n}{2} - 1$. It leads to m graphs by setting $t_i = 2$ for $1 \leq i \leq m$. Thus, there are $\frac{n}{2}$ threshold graphs with distinct distance eigenvalues. The case of odd n is similar. We present such graphs in Table 1 for $n = 5, 6, 7, 8, 9$.

Table 2
The distance spectrum and distance energy of threshold graphs on 8 vertices.

$(s_1, t_1, \dots, s_m, t_m)$	Distance spectrum	Distance energy
$(4, 4)$	$\{8.77, 0.23, [-1]^3, [-2]^3\}$	18
$(2, 2, 2, 2)$	$\{9.82, -0.26, [-1]^2, [-2]^3, -3.05\}$	19.64
$(2, 1, 2, 3)$	$\{9.30, 0, -0.65, [-1]^2, [-2]^2, -2.65\}$	18.60
$(2, 1, 3, 2)$	$\{10.62, -0.22, -0.63, -1, [-2]^3, -2.78\}$	21.25
$(2, 2, 1, 3)$	$\{8.44, -0.15, -0.56, [-1]^3, -2, -2.72\}$	16.88
$(2, 2, 3, 1)$	$\{11.31, -0.45, -0.61, -1, [-2]^3, -3.25\}$	22.61
$(2, 3, 1, 2)$	$\{8.73, -0.29, -0.54, [-1]^3, -2, -2.90\}$	17.46
$(2, 3, 2, 1)$	$\{10.32, -0.37, -0.64, [-1]^2, [-2]^2, -3.31\}$	20.64
$(1, 2, 2, 3)$	$\{9.02, -0.08, [-1]^4, -2, -2.94\}$	18.04
$(1, 2, 3, 2)$	$\{10.36, -0.25, [-1]^3, [-2]^2, -3.11\}$	20.71
$(1, 3, 2, 2)$	$\{9.56, -0.32, [-1]^4, -2, -3.24\}$	19.12
$(3, 1, 2, 2)$	$\{10.36, -0.13, -0.52, -1, [-2]^3, -2.70\}$	20.73
$(3, 2, 1, 2)$	$\{9.29, 0, -0.55, [-1]^2, [-2]^2, -2.73\}$	18.57
$(3, 2, 2, 1)$	$\{10.82, -0.09, -0.64, -1, [-2]^3, -3.08\}$	21.64
$(1, 1, 3, 3)$	$\{9.59, 0.04, [-1]^3, [-2]^2, -2.63\}$	19.27
$(1, 3, 1, 3)$	$\{8.17, -0.30, [-1]^5, -2.87\}$	16.33
$(1, 3, 3, 1)$	$\{11.07, -0.60, [-1]^3, [-2]^2, -3.47\}$	22.14
$(3, 1, 3, 1)$	$\{11.81, -0.34, -0.62, [-2]^4, -2.85\}$	23.61
$(3, 1, 1, 3)$	$\{9.03, 0.07, -0.62, [-1]^2, [-2]^2, -2.48\}$	18.20
$(3, 3, 1, 1)$	$\{9.55, -0.05, -0.71, [-1]^2, [-2]^2, -2.89\}$	19.20
$(1, 1, 3, 1, 1, 1)$	$\{11.10, -0.20, -0.75, -1, [-2]^2, -2.42, -2.73\}$	22.19
$(1, 1, 1, 3, 1, 1)$	$\{9.28, -0.24, -0.71, [-1]^3, -2.37, -2.96\}$	18.56
$(1, 1, 1, 1, 1, 3)$	$\{8.73, -0.08, -0.69, [-1]^3, -2.27, -2.69\}$	17.45
$(1, 1, 1, 1, 3, 1)$	$\{11.55, -0.51, -0.64, -1, [-2]^2, -2.30, -3.10\}$	23.10
$(1, 3, 1, 1, 1, 1)$	$\{9.81, -0.48, -0.76, [-1]^3, -2.32, -3.25\}$	19.62
$(3, 1, 1, 1, 1, 1)$	$\{10.58, -0.09, -0.64, -0.77, [-2]^2, -2.28, -2.80\}$	21.16
$(1, 1, 1, 2, 1, 2)$	$\{9.00, -0.23, -0.58, [-1]^3, -2.34, -2.85\}$	18.00
$(1, 1, 1, 2, 2, 1)$	$\{10.56, -0.35, -0.65, [-1]^2, -2, -2.35, -3.22\}$	21.13
$(1, 1, 2, 1, 1, 2)$	$\{9.83, -0.10, -0.65, [-1]^2, -2, -2.37, -2.71\}$	19.66
$(1, 1, 2, 1, 2, 1)$	$\{11.31, -0.30, -0.66, -1, [-2]^2, -2.43, -2.92\}$	22.63
$(1, 2, 1, 2, 1, 1)$	$\{9.54, -0.31, -0.72, [-1]^3, -2.46, -3.05\}$	19.08
$(1, 2, 2, 1, 1, 1)$	$\{10.58, -0.30, -0.75, [-1]^2, -2, -2.40, -3.12\}$	21.15
$(1, 2, 1, 1, 1, 2)$	$\{9.28, -0.27, -0.70, [-1]^3, -2.31, -3.00\}$	18.56
$(1, 2, 1, 1, 2, 1)$	$\{10.81, -0.49, -0.68, [-1]^2, -2, -2.37, -3.28\}$	21.63
$(2, 1, 2, 1, 1, 1)$	$\{10.38, -0.22, -0.65, -0.75, [-2]^2, -2.38, -2.83\}$	21.66
$(2, 1, 1, 2, 1, 1)$	$\{9.80, -0.12, -0.69, -0.74, -1, -2, -2.37, -2.88\}$	19.60
$(2, 1, 1, 1, 1, 2)$	$\{9.55, -0.15, -0.65, -0.76, -1, -2, -2.28, -2.77\}$	19.10
$(2, 1, 1, 1, 2, 1)$	$\{11.06, -0.32, -0.64, -0.74, [-2]^2, -2.31, -3.04\}$	22.12
$(2, 2, 1, 1, 1, 1)$	$\{10.06, -0.31, -0.59, -0.76, -1, -2, -2.31, -3.08\}$	20.11
$(1, 1, 1, 1, 2, 2)$	$\{10.08, -0.23, -0.62, [-1]^2, -2, -2.29, -2.93\}$	20.17
$(1, 1, 2, 2, 1, 1)$	$\{10.07, -0.04, -0.72, [-1]^2, -2, -2.48, -2.83\}$	20.13
$(1, 1, 1, 1, 1, 1, 1, 1)$	$\{10.31, -0.27, -0.68, -0.78, -1, -2.24, -2.37, -2.98\}$	20.63

For a square matrix M the triple $(n_-(M), n_0(M), n_+(M))$ is called the inertia M, where $n_-(M)$ and $n_+(M)$ denote the number of negative and positive eigenvalues, respectively, whereas, $n_0(M)$ is the nullity of M. By Theorem 3.1, we get the inertia of $\mathcal{D}(\Gamma) + 2I$.

Corollary 3.4. *Let Γ be a threshold graph with representation sequence $(0^{s_1}, 1^{t_1}, \ldots, 0^{s_m}, 1^{t_m})$ and $s = s_1 + \cdots + s_m$, $t = t_1 + \cdots + t_m$. If $\mathcal{D}(\Gamma)$ is the distance matrix of Γ, then the inertia of $\mathcal{D}(\Gamma) + 2I$ is $(m-1, s-m, t+1)$.*

A graph Γ is said to be *determined by its distance spectrum* if there is no graph $\Gamma' \ncong \Gamma$ sharing the same distance spectrum with Γ. Note that the threshold graph Γ with representation sequence $(0^{s_1}, 1^{t_1})$ is a complete multipartite graph, i.e., $\Gamma \cong K_{s_1,1,\ldots,1}$. In 2014, Jin and Zhang [11] proved that complete multipartite graphs are determined by their distance spectra. However, there is no result about whether threshold graphs are determined by their distance spectra. Thus, we propose the following result.

Problem 1. Whether threshold graphs are determined by their distance spectra? Or more special, whether two different threshold graphs can share the same distance spectrum?

The *distance energy* of a graph Γ is the sum of the absolute values of its distance eigenvalues. This concept was introduced by Indulal, Gutman and Vijayakumar [10] and obtains widely attentions [9,16,22]. Thus, we are interested in the distance energy of threshold graphs.

Problem 2. Among all threshold graphs of order n, which ones have the largest distance energy? Note that a threshold graph has a representation sequence, say $(0^{s_1}, 1^{t_1}, \ldots, 0^{s_m}, 1^{t_m})$, where $s_i, t_i \geq 1$ for $1 \leq i \leq m$ and $s_1 + \cdots + s_m + t_1 + \cdots + t_m = n$. For given n and m, which threshold graph has the largest distance energy?

We end up our paper by calculating the distance spectra and distance energies of all threshold graphs on 8 vertices in Table 2, which may be helpful for solving the problems above.

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References

[1] M. Aouchiche, P. Hansen, Distance spectra of graphs: a survey, Linear Algebra Appl. 458 (2014) 301–384.
[2] R.B. Bapat, On the adjacency matrix of a threshold graph, Linear Algebra Appl. 439 (2013) 3008–3015.
[3] A. Banerjeea, R. Mehataria, On the normalized spectrum of threshold graphs, Linear Algebra Appl. 530 (2017) 288–304.
[4] A.E. Brouwer, W.H. Haemers, Spectra of Graphs, Springer, Heidelberg, 2012.
[5] C.D. Godsil, G. Royle, Algebraic Graph Theory, Springer-Verlag, Berlin, 2001.
[6] R.L. Graham, H.O. Pollack, On the addressing problem for loop switching, Bell Syst. Tech. J. 50 (1971) 2495–2519.

[7] F. Harary, A.J. Schwenk, Which graphs have integral spectra?, in: R. Bari, F. Harary (Eds.), Graphs and Combinatorics, in: Lecture Notes in Math., vol. 406, Springer-Verlag, Berlin, 1974, pp. 45–51.
[8] P.B. Henderson, Y. Zalcstein, A graph-theoretic characterization of the PV class of synchronizing primitives, SIAM J. Comput. 6 (1977) 88–108.
[9] G. Indulal, Sharp bounds on the distance spectral radius and the distance energy of graphs, Linear Algebra Appl. 430 (2009) 106–113.
[10] G. Indulal, I. Gutman, A. Vijayakumar, On distance energy of graphs, MATCH Commun. Math. Comput. Chem. 60 (2008) 461–472.
[11] Y.L. Jin, X.D. Zhang, Complete multipartite graphs are determined by their distance spectra, Linear Algebra Appl. 448 (2014) 285–291.
[12] D.P. Jacobs, V. Trevisan, F. Tura, Eigenvalue location in threshold graphs, Linear Algebra Appl. 439 (2013) 2762–2773.
[13] D.P. Jacobs, V. Trevisan, F. Tura, Distance eigenvalue location in threshold graphs, in: Proc. of Workshop on Distance Geometry Appl., 2013, pp. 157–161.
[14] D.P. Jacobs, V. Trevisan, F. Tura, Computing the characteristic polynomial of threshold graphs, J. Graph Algorithms Appl. 18 (2014) 709–719.
[15] D.P. Jacobs, V. Trevisan, F. Tura, Eigenvalues and energy in threshold graphs, Linear Algebra Appl. 465 (2015) 412–425.
[16] H.Q. Lin, R.F. Liu, X.W. Lu, The inertia and energy of the distance matrix of a connected graph, Linear Algebra Appl. 407 (2015) 29–39.
[17] Z.Z. Lou, Q.X. Huang, X.Y. Huang, Construction of graphs with distinct eigenvalues, Discrete Math. 340 (2017) 607–616.
[18] L. Lu, Q.X. Huang, X.Y. Huang, The graphs with exactly two distance eigenvalues different from -1 and -3, J. Algebraic Combin. 45 (2017) 629–647.
[19] V.N. Mahadev, U.N. Peled, Threshold Graphs and Related Topics, Elsevier, 1995.
[20] J.S. do Nascimento, Maria Aguieiras, A. de Freitas, Renata R. Del-Vecchio, Matriz distância de grafos threshold, Proc. Ser. Braz. Soc. Appl. Comput. Math. 6 (2018).
[21] I. Sciriha, S. Farrugia, On the spectrum of threshold graphs, ISRN Discrete Math. 2011 (2011), https://doi.org/10.5402/2011/108509.
[22] D. Stevanović, G. Indulal, The distance spectrum and energy of the compositions of regular graphs, Appl. Math. Lett. 22 (2009) 1136–1140.