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On graphs whose smallest distance (signless Laplacian) eigenvalue has large multiplicity

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ABSTRACT

Denote by $\partial_1 \geq \partial_2 \geq \cdots \geq \partial_n$ (resp. $\partial_1^Q \geq \partial_2^Q \geq \cdots \geq \partial_n^Q$) the distance (resp. distance signless Laplacian) eigenvalues of a connected graph G on n vertices, and by $m(\partial_i)$ (resp. $m(\partial_i^Q)$) the multiplicity of ∂_i (resp. ∂_i^Q). In this paper, we completely determine the graphs with $m(\partial_n) = n - 3$ and $m(\partial_n^Q) = n - 2$, respectively.

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1. Introduction

Let $G = (V, E)$ be a connected simple graph with vertex set $V = \{v_1, v_2, \dots, v_n\}$ and edge set $E = \{e_1, e_2, \dots, e_m\}$. We denoted by $d_G(v_i, v_j)$ the *distance* between v_i and v_j, which is defined as the length of a shortest path between them. The *diameter* of G, denoted by $d(G)$, is the maximum distance between any two vertices of G. The *distance matrix* of G, denoted by $\mathcal{D}(G)$, is the $n \times n$ matrix whose (i,j)-entry is equal to $d_G(v_i, v_j)$, for $1 \leq i, j \leq n$. The *transmission* $Tr(v_i)$ of a vertex v_i is defined as the sum of the distances between v_i and all other vertices in G, that is, $Tr(v_i) = \sum_{j=1}^{n} d_G(v_i, v_j)$. For more details about the distance matrix, we refer the reader to [1]. As the distance matrix $\mathcal{D}(G)$ is symmetric, all of its eigenvalues are real, we can list them as $\partial_1(G) \geq \partial_2(G) \geq \cdots \geq \partial_n(G)$. Denote by $m(\partial_i(G))$ the *multiplicity* of $\partial_i(G)$. The multi-set of these eigenvalues is called the *distance spectrum* of G, denoted by $\mathrm{Spec}_{\mathcal{D}}(G)$. Aouchiche and Hansen [2] introduced the signless Laplacian for the distance matrix of G as $\mathcal{D}^Q(G) = Tr(G) + \mathcal{D}(G)$, where $Tr(G) = \mathrm{diag}(Tr(v_1), Tr(v_2), \dots, Tr(v_n))$ is the diagonal matrix of the vertex transmissions in G. Also, the eigenvalues of $\mathcal{D}^Q(G)$ can be arranged as $\partial_1^Q(G) \geq \partial_2^Q(G) \geq \cdots \geq \partial_n^Q(G)$, and $m(\partial_i^Q(G))$ denotes the multiplicity of $\partial_i^Q(G)$. Further, the *distance signless Laplacian spectrum* of G, denoted by $\mathrm{Spec}_Q(G)$, could be similarly defined.

The graphs with few distinct adjacency eigenvalues always possess nice combinatorial properties and form an interesting class to study. Many mathematicians have been attracted to this topic, and they obtain many beautiful results, see [3–9]. However, it seems to be

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difficult to completely determine such graphs. Even the graphs with exactly three distinct eigenvalues have not been completely determined up to now. Naturally, mathematicians turn their eyes to some special classes of such graphs. Recently, Cioabă et al. determined the graphs with all but two eigenvalues equal to ± 1 [10], and 0 or -2 [11]. With respect to distance matrix and its Laplacians, there are a few of results on this subject. For instance, Lin et al. [12] and Celso et al. [13] independently determined the graphs whose largest distance Laplacian eigenvalue has multiplicity $n-2$; Lu et al. completely determined the graphs whose largest distance Laplacian eigenvalue has multiplicity $n-3$ [14] and the graphs with exactly two distance eigenvalues different from -1 and -3 [15]; Koolen et al. [16] proved that the graphs whose smallest distance eigenvalue has multiplicity $n-2$ are complete bipartite graphs. Motivated by these works, in this paper, we completely determine those graphs whose smallest distance eigenvalue (resp. distance signless Laplacian eigenvalue) has multiplicity $n-3$ (resp. $n-2$).

At the end of this part, we introduce some general definitions and notions. Let G be a connected graph. We always denote by $N_G(v)$ the *neighborhood* of v in G, that is, $N_G(v) = \{u \in V(G) \mid u \sim v\}$. Denote by $\bar{G}$ the *complement* of G, which is the graph with vertex set $V(\bar{G}) = V(G)$ and two vertices are adjacent whenever they are not adjacent in G. For a subset $S \subseteq V(G)$, the subgraph of G induced by S is denoted by $G[S]$. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two connected graphs, the (disjoint) *union* of G_1 and G_2 is the graph $G_1 \cup G_2$, whose vertex set is $V_1 \cup V_2$ and edge set is $E_1 \cup E_2$, and we write mG for the disjoint union of m copies of G. The *join* of G_1 and G_2 is the graph $G_1 \nabla G_2$, which is obtained from $G_1 \cup G_2$ by joining each vertex of G_1 with every vertex of G_2. When it is clear from the context which graph G we mean, we delete G from the notations like $d_G(v_i, v_j)$, $N_G(v)$, $\partial_i(G)$, and so on. Besides, we always write, respectively, K_n, P_n and C_n for the complete graph, the path and the cycle on n vertices. For integers $a_1, a_2, \ldots, a_k \geq 1$, let $K_{a_1,a_2,\ldots,a_k}$ denote the complete k-partite graph on $a_1 + a_2 + \cdots + a_k$ vertices. For two positive integers s, t, the graph $sK_1 \nabla (K_t \cup K_1)$ is always denoted by $J(s,t)$ and we present $J(1,2)$, $J(3,5)$ and $J(s,t)$ in Figure 2. All other notations not mentioned here are standard in [17].

2. Preliminaries

In this section, we shall list some known results that will be needed in the next section.

Theorem 2.1 [18]: *Let A be a real symmetric matrix of order n with eigenvalues $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$ and let M be a principal submatrix of A with order $m \leq n$ and eigenvalues $\lambda_1(M) \geq \lambda_2(M) \geq \cdots \geq \lambda_m(M)$. Then $\lambda_i(A) \geq \lambda_i(M) \geq \lambda_{n-m+i}(A)$ for all $1 \leq i \leq m$.*

Let H be an induced subgraph of G. If $d(H) \leq 2$, then $\mathcal{D}(H)$ is a principal submatrix of $\mathcal{D}(G)$. We immediately get the following result from Theorem 2.1.

Corollary 2.1: *Let G be a connected graph on n vertices and H an induced subgraph of G on m vertices with $d(H) \leq 2$. Then $\partial_{n-m+i}(G) \leq \partial_i(H) \leq \partial_i(G)$ for all $1 \leq i \leq m$.*

Lemma 2.1 [17], Theorem 1.3.6: *Let M be an irreducible symmetric matrix with non-negative entries. Then the largest eigenvalue λ_1 of M is simple.*

A graph G is said to be a *cograph* if it contains no induced P_4. There's a pretty result about cographs.

Lemma 2.2 [19]: *Given a graph G, the following statements are equivalent:*

(i) *G is a cograph.*
(ii) *The complement of any connected subgraph of G with at least two vertices is disconnected.*
(iii) *Every connected induced subgraph of G has diameter less than or equal to* 2.

From Lemma 2.2(ii), it is easy to see that a connected cograph must be the join of some graphs because its complement is not connected.

Lemma 2.3 [16]: *Let G be a connected graph with $n \geq 2$ vertices, distance matrix $\mathcal{D}$ and diameter d. Let ∂_n be the smallest distance eigenvalue of G with multiplicity $m(\partial_n)$. Then, the following results hold:*

(i) *$\partial_n = -2$ if and only if G is a complete s-partite graph $K_{a_1,a_2,\ldots,a_s}$, where $1 \leq a_1 \leq \cdots \leq a_s$ are integers such that $a_s \geq 2$ and $s = n - m(\partial_n)$.*
(ii) *$\partial_n = -1$ if and only if $G = K_n$.*

3. Main results

In this section, we will give our main results. At the beginning of this part, we introduce some additional marks. Let $\mathcal{G}(n)$ be the set of connected graphs on n vertices. We denote by

$$\mathcal{G}_{\mathcal{D}}(n,k) = \{G \in \mathcal{G}(n) \mid m(\partial_n) = k\} \text{ and } \mathcal{G}_{\mathcal{Q}}(n,k) = \{G \in \mathcal{G}(n) \mid m(\partial_n^Q) = k\}.$$

In what follows, we focus on characterizing the graphs in $\mathcal{G}_{\mathcal{D}}(n, n-3)$ and $\mathcal{G}_{\mathcal{Q}}(n, n-2)$.

Lemma 3.1: *Let A be an irreducible symmetric matrix of order $n > 2$ with non-negative integral entries. If λ is an eigenvalue of A with multiplicity $n-k$ and $n \geq 2k$, then λ is integral.*

Proof: Let $f(x)$ be the characteristic polynomial of A. As A only contains integral entries, we obtain that $f(x)$ is a monic polynomial with integral coefficients. Let $p(x)$ be the minimal polynomial of λ, then $p(x) \in \mathbb{Z}[x]$ is irreducible in $\mathbb{Q}[x]$ and $(p(x))^{n-k} | f(x)$. If $p(x)$ is of degree 1, then $p(x) = x - \lambda$ and so λ is an integer. Otherwise, we assume that $p(x)$ has degree at least 2. Therefore, $p(x)$ has another root λ', which is also an eigenvalue of A with multiplicity $n-k > 1$. Note that $\rho \neq \lambda, \lambda'$, where ρ is the largest eigenvalue of A which is simple by Lemma 2.1. This implies that $2(n-k) \leq n-1$, a contradiction. Thus, the result follows. □

The following two results are immediate by applying Lemma 3.1 to the distance matrix and distance signless Laplacian matrix, respectively.

Corollary 3.1: *If $G \in \mathcal{G}_{\mathcal{D}}(n, n-k)$ with $n \geq 2k > 2$, then ∂_n is integral.*

Corollary 3.2: *If $G \in \mathcal{G}_{\mathcal{Q}}(n, n-k)$ with $n \geq 2k > 2$, then ∂_n^Q is integral.*

The following result is a characterization of the complete multipartite graphs.

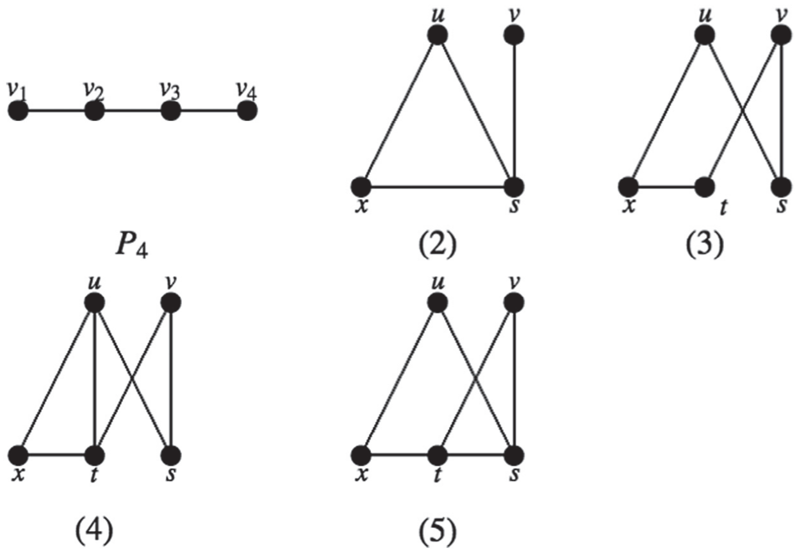

Figure 1. The graphs used in Lemma 3.2.

Lemma 3.2: *Let G be a connected graph with diameter $d(G) = 2$. If G contains neither induced C_5 nor induced $J(1, 2)$, then G is a complete multipartite graph.*

***Proof*:** It suffices to prove that any pair of non-adjacent vertices in G must have the same neighborhood. Let $u, v \in V(G)$ and $u \nsim v$. Since $d(G) = 2$, there exists $s \in V(G)$ such that $s \sim u$ and $s \sim v$. By the symmetry of u and v, we only need to show that $x \sim u$ implies $x \sim v$ for any $x \in V(G)$ $(x \neq s)$.

Suppose on the contrary that $x \nsim v$. If $x \sim s$, then G contains the induced subgraph $G[u, x, s, v] = J(1, 2)$ (see Figure 1(2)), a contradiction. If $x \nsim s$, then there exists $t \in V(G)$ such that $t \sim x, v$ because $d(G) = 2$. Therefore, when $t \nsim u, s$, we get the induced subgraph $G[u, x, t, s, v] = C_5$ (see Figure 1(3)), a contradiction; when $t \sim u$, we get the induced subgraph $G[u, x, t, v] = J(1, 2)$ (see Figure 1(4)), a contradiction; when $t \sim s$, we get the induced subgraph $G[x, t, s, v] = J(1, 2)$ (see Figure 1(5)), a contradiction. □

Remark 1: *This result has been noticed by Koolen [16] and we formally state it here.*

Using above notations and tools, in what follows, we will characterize the graphs belonging to $\mathcal{G}_{\mathcal{D}}(n, n-3)$ and $\mathcal{G}_{\mathcal{Q}}(n, n-2)$ in Sections 3.1 and 3.2, respectively.

3.1. Characterization of graphs in $\mathcal{G}_{\mathcal{D}}(n, n-3)$

At the beginning, we give some forbidden subgraphs for graphs in $\mathcal{G}_{\mathcal{D}}(n, n-3)$.

Lemma 3.3: *If $G \in \mathcal{G}_{\mathcal{D}}(n, n-3)$ with $n \geq 6$, then none of P_4, C_5 or $J(1, 2)$ can be an induced subgraph of G.*

***Proof*:** We will prove our result by contradiction.

If $P_4 = v_1v_2v_3v_4$ (see Figure 2) is an induced subgraph of G. It is easy to see that either $d_G(v_1, v_4) = d_{P_4}(v_1, v_4) = 3$ or $d_G(v_1, v_4) = d_{P_4}(v_1, v_4) - 1 = 2$. Then either M or M' is a principal submatrix of $D(G)$ with respect to P_4, where

$$M = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}, M' = \begin{pmatrix} 0 & 1 & 2 & 2 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 2 & 2 & 1 & 0 \end{pmatrix}.$$

By Theorem 2.1, we have $\partial_n(G) \leq \lambda_4(M) \leq \partial_4(G)$ or $\partial_n(G) \leq \lambda_4(M') \leq \partial_4(G)$. Note that $\partial_n(G) = \partial_4(G)$. We have that either $\partial_n(G) = \lambda_4(M) \approx -3.41$ or $\partial_n(G) = \lambda_4(M') \approx -2.62$. However, by Corollary 3.1, we get $\partial_n(G) \in \mathbb{Z}$, a contradiction.

If C_5 is an induced subgraph of G, by Corollary 2.1, we have $\partial_n(G) = \partial_{n-1}(G) \leq \partial_4(C_5) \leq \partial_4(G) = \partial_n(G)$. Thus $\partial_n(G) = \partial_4(C_5) \approx -2.62$, which is also impossible.

Similarly, we conclude that G contains no induced $J(1,2)$ because $\partial_4(J(1,2)) \approx -2.38 \notin \mathbb{Z}$. □

Combining Lemmas 3.2 and 3.3, we get the first main result.

Theorem 3.1: *For any integer $n \geq 6$, we have*

$$\mathcal{G}_{\mathcal{D}}(n, n-3) = \{K_{n_1,n_2,n_3} \mid 1 \leq n_1 \leq n_2 \leq n_3, n_1 + n_2 + n_3 = n\}.$$

***Proof*:** Let $G \in \mathcal{G}_{\mathcal{D}}(n, n-3)$. From Lemma 3.3, G contains no induced P_4 and thus $d(G) \leq 2$. Note that $K_n \notin \mathcal{G}_{\mathcal{D}}(n, n-3)$. We get $d(G) = 2$. By Lemma 3.3, G contains neither induced C_5 nor induced $J(1,2)$. Thus, from Lemma 3.2, G is complete multipartite. By Lemma 2.3(i), we have $\partial_n = -2$ and $G = K_{n_1,n_2,n_3}$, where $1 \leq n_1 \leq n_2 \leq n_3$ and $n_1 + n_2 + n_3 = n$. Conversely, by Lemma 2.3(i) again, for any integers $1 \leq n_1 \leq n_2 \leq n_3$ and $n_1 + n_2 + n_3 = n$, we obtain $\partial_n(K_{n_1,n_2,n_3}) = -2$ and $m(\partial_n(K_{n_1,n_2,n_3})) = n - 3$. Thus, $K_{n_1,n_2,n_3} \in \mathcal{G}_{\mathcal{D}}(n, n-3)$. □

From Lemma 2.3(i), we see that $K_{n_1,\ldots,n_k} \in \mathcal{G}_{\mathcal{D}}(n, n-k)$ for any $k < n$. Moreover, Theorem 3.1 implies that $\mathcal{G}_{\mathcal{D}}(n, n-3) = \{K_{n_1,n_2,n_3}\}$ whenever $n \geq 6$. Accordingly, we propose the following problem.

Problem 1: For $n \geq 2k > 2$, are $K_{n_1,\ldots,n_k}$ the only graphs in $\mathcal{G}_{\mathcal{D}}(n, n-k)$? From computational experiments, we find that K_{n_1,n_2,n_3,n_4} are the only graphs in $\mathcal{G}_{\mathcal{D}}(n, n-4)$ for $n = 7$ and $n = 8$.

3.2. *Characterization of graphs in $\mathcal{G}_{\mathcal{Q}}(n, n-2)$*

First of all, we give an upper bound of $\partial_n^Q(G)$ for the connected graph G.

Lemma 3.4: *Let G be a connected graph on $n \geq 2$ vertices. Then $\partial_n^Q < Tr(v)$ for any $v \in V(G)$.*

***Proof*:** By the famous Rayleigh quotient, we have $\partial_n^Q \leq \frac{x^T \mathcal{D}^Q x}{x^T x}$ for any $0 \neq x \in \mathbb{R}^n$. Taking $x = e_v$, whose vth entry is 1 and all other entries are 0, we get that $\partial_n^Q \leq \frac{e_v^T \mathcal{D}^Q e_v}{e_v^T e_v} = Tr(v)$. If the equality holds, then e_v is an eigenvector of $\mathcal{D}^Q$ with respect to ∂_n^Q. It leads to that v is an isolate vertex, which is impossible. □

Now we give the following basic result.

Theorem 3.2: *For an integer $n \geq 2$, we have*

$$\mathcal{G}_Q(n, n-1) = \{K_n\}.$$

***Proof*:** Obviously, $K_n \in \mathcal{G}_Q(n, n-1)$. Conversely, suppose that $G \in \mathcal{G}_Q(n, n-1)$. It suffices to show that $Tr(v) = n-1$ for each $v \in V(G)$. Let $M = D^Q(G) - \partial_n^Q I$, then the rank of M is 1. Denote by $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let $v_i v_j$ be an edge of G. We consider the principal submatrix of M induced by v_i and v_j, which is

$$M(v_i, v_j) = \begin{pmatrix} Tr(v_i) - \partial_n^Q & 1 \\ 1 & Tr(v_j) - \partial_n^Q \end{pmatrix}.$$

Since $\mathrm{rank}(M) = 1$, we have $\det(M(v_i, v_j)) = 0$. It follows that $(Tr(v_i) - \partial_n^Q)(Tr(v_j) - \partial_n^Q) = 1$. By Lemma 3.4 and Corollary 3.2, $Tr(v) - \partial_n^Q$ is a positive integer for any $v \in V(G)$. This implies that $Tr(v_i) - \partial_n^Q = Tr(v_j) - \partial_n^Q = 1$, that is, $Tr(v_i) = Tr(v_j) = \partial_n^Q + 1$. Since G is connected, we claim that $Tr(v) = \partial_n^Q + 1$ for any $v \in V(G)$. Hence we only need to prove $Tr(v_1) = n-1$. Suppose on the contrary that $Tr(v_1) = Tr(v) > n-1$. Then there exists $v_k \not\sim v_1$, that is, $d(v_1, v_k) = a > 1$. We consider the principal submatrix of M induced by v_1 and v_k, which is

$$M(v_1, v_k) = \begin{pmatrix} Tr(v_1) - \partial_n^Q & a \\ a & Tr(v_k) - \partial_n^Q \end{pmatrix} = \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}.$$

Then $\det(M(v_1, v_k)) = 1 - a^2 \neq 0$, contrary to $\mathrm{rank}(M) = 1$. □

For any graph $G \in \mathcal{G}_Q(n, n-2)$, we know that $M = \mathcal{D}^Q(G) - \partial_n^Q I$ has rank 2. It means that any principal submatrix of M with order greater than 2 has determinant 0. In what follows, we always denote by $m(v)$ the (v, v)-entry of M, that is, $m(v) = Tr(v) - \partial_n^Q$. From Corollary 3.2 and Lemma 3.4, we see that $m(v)$ is a positive integer for any $v \in V(G)$. Now we give the following useful result.

Lemma 3.5: *Let $G \in \mathcal{G}_Q(n, n-2)$ with $n \geq 4$ and $M = \mathcal{D}^Q(G) - \partial_n^Q I$. Let $P_3 = v_1 v_2 v_3$ be an induced path of G. Then we have*

(i) $m(v_1) = m(v_3) = 2$ *and* $m(v_2) \geq 1$, *or*
(ii) $m(v_1) = 1$, $m(v_3) = 5$ *and* $m(v_2) = 2$, *or*
(iii) $m(v_1) = 5$, $m(v_3) = 1$ *and* $m(v_2) = 2$.

***Proof*:** We consider the principal submatrix of M induced by P_3, which is

$$M[P_3] = \begin{pmatrix} m(v_1) & 1 & 2 \\ 1 & m(v_2) & 1 \\ 2 & 1 & m(v_3) \end{pmatrix}.$$

Note that $m(v_1)$, $m(v_2)$ and $m(v_3)$ are positive integers. Since $\det(M[P_3]) = 0$, we have $m(v_1)m(v_2)m(v_3) + 4 = m(v_1) + 4m(v_2) + m(v_3)$, which gives that

$$(m(v_1)m(v_3) - 4)m(v_2) = m(v_1) + m(v_3) - 4. \tag{1}$$

If $m(v_1)m(v_3) = 4$, then from (1) we deduce that $m(v_1) = m(v_3) = 2$, and $m(v_2)$ is an arbitrary positive integer.

If $m(v_1)m(v_3) \neq 4$, then we have

$$m(v_2) = \frac{m(v_1) + m(v_3) - 4}{m(v_1)m(v_3) - 4}.$$

Note that $m(v_2) \geq 1$ is integral. Since $m(v_1)m(v_3) > m(v_1) + m(v_3)$ whenever $m(v_1)$, $m(v_3) \geq 2$ and $m(v_1)m(v_3) \neq 4$, at least one of $m(v_1)$ and $m(v_3)$ is equal to 1. If $m(v_1) = 1$, we have

$$m(v_2) = \frac{m(v_3) - 3}{m(v_3) - 4} = 1 + \frac{1}{m(v_3) - 4}.$$

Since $m(v_2) \geq 1$ is integral, we obtain $m(v_3) = 5$ and $m(v_2) = 2$. Similarly, if $m(v_3) = 1$, then $m(v_1) = 5$ and $m(v_2) = 2$. □

Remark 2: From the conclusion of Lemma 3.5, it is easy to see that

(i) if $m(v_2) \neq 2$ then $m(v_1) = m(v_3) = 2$;
(ii) $m(v_1) = 2$ if and only if $m(v_3) = 2$;
(iii) $m(v_1) = 1$ (resp. $m(v_1) = 5$) if and only if $m(v_3) = 5$ (resp. $m(v_3) = 1$).

Using Remark 2, we now give some forbidden subgraphs of the graphs in $\mathcal{G}_Q(n, n-2)$.

Lemma 3.6: *If $G \in \mathcal{G}_Q(n, n-2)$ with $n \geq 4$, then neither P_4 nor C_5 could be an induced subgraph of G.*

***Proof*:** Recall that the matrix $M = \mathcal{D}^Q - \partial_n^Q I$ has rank 2. The determinant of any principal submatrix of M with order 3 is equal to 0. We will use this fact frequently.

First assume that $P_4 = v_1v_2v_3v_4$ (see Figure 2) is an induced subgraph of G. Note that either $d_G(v_1, v_4) = d_{P_4}(v_1, v_4) = 3$ or $d_G(v_1, v_4) = d_{P_4}(v_1, v_4) - 1 = 2$. Then the principal submatrix of M induced by v_1, v_2, v_3, v_4 is given by

$$M[P_4] = \begin{pmatrix} m(v_1) & 1 & 2 & 3 \\ 1 & m(v_2) & 1 & 2 \\ 2 & 1 & m(v_3) & 1 \\ 3 & 2 & 1 & m(v_4) \end{pmatrix} \text{ or } M'[P_4] = \begin{pmatrix} m(v_1) & 1 & 2 & 2 \\ 1 & m(v_2) & 1 & 2 \\ 2 & 1 & m(v_3) & 1 \\ 2 & 2 & 1 & m(v_4) \end{pmatrix}$$

where $m(v_i) = Tr(v_i) - \partial_n^Q$ are positive integers. We consider the following two cases.

Case 1.1. $m(v_2) \neq 2$;

Note that $v_1v_2v_3$ is an induced P_3 of G. By Remark 2 (i), we have $m(v_1) = m(v_3) = 2$. Then the principal submatrices of $M[P_4]$ and $M'[P_4]$ induced by v_1, v_3, v_4 are given by

$$M_1 = \begin{pmatrix} 2 & 2 & 3 \\ 2 & 2 & 1 \\ 3 & 1 & m(v_4) \end{pmatrix}, M_2 = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & m(v_4) \end{pmatrix}.$$

We obtain a contradiction because $\det(M_1) = -8 \neq 0$ and $\det(M_2) = -2 \neq 0$.

Case 1.2. $m(v_2) = 2$.

Note that $v_2v_3v_4$ is an induced P_3 of G. By Remark 2 (ii), we get $m(v_4) = 2$. Then the principal submatrices of $M[P_4]$ and $M'[P_4]$ induced by v_1, v_2, v_4 are given by

$$M_3 = \begin{pmatrix} m(v_1) & 1 & 3 \\ 1 & 2 & 2 \\ 3 & 2 & 2 \end{pmatrix}, M_4 = \begin{pmatrix} m(v_1) & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix},$$

which are impossible because $\det(M_3) = -8 \neq 0$ and $\det(M_4) = -2 \neq 0$.

Now assume that $C_5 = v_1v_2v_3v_4v_5$ (see Figure 2) is an induced subgraph of G. Then the principal submatrix of M induced by v_1, v_2, v_3, v_4, v_5 are

$$M[C_5] = \begin{pmatrix} m(v_1) & 1 & 2 & 2 & 1 \\ 1 & m(v_2) & 1 & 2 & 2 \\ 2 & 1 & m(v_3) & 1 & 2 \\ 2 & 2 & 1 & m(v_4) & 1 \\ 1 & 2 & 2 & 1 & m(v_5) \end{pmatrix}.$$

We consider the following two cases.

Case 2.1. $m(v_2) \neq 2$;

Note that $v_1v_2v_3$ is an induced P_3 of G. By Remark 2 (i), we have $m(v_1) = m(v_3) = 2$. Since $v_3v_4v_5$ is an induced P_3 and $m(v_3) = 2$, by Remark 2 (ii), we have $m(v_5) = 2$. Similarly, since $v_5v_1v_2$ is an induced P_3 and $m(v_5) = 2$, we obtain $m(v_2) = 2$, which contradicts the assumption.

Case 2.2. $m(v_2) = 2$.

Since $v_2v_3v_4$ is an induced P_3 of G and $m(v_2) = 2$, by Remark 2 (ii), we have $m(v_4) = 2$. Also, since $v_2v_1v_5$ is an induced P_3 and $m(v_2) = 2$, we get $m(v_5) = 2$. Then the principal submatrix of $M[C_5]$ induced by v_2, v_4, v_5 is given by

$$M_5 = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix},$$

which is impossible due to $\det(M_5) = -2 \neq 0$. □

As in Lemma 3.5, by considering the induced subgraph K_3, we have the following result.

Lemma 3.7: *Let $G \in \mathcal{G}_Q(n, n-2)$ with $n \geq 4$ and $M = \mathcal{D}^Q(G) - \partial_n^Q I$. Let K_3 be an induced subgraph of G with vertex set $V = \{v_1, v_2, v_3\}$. Then, at least two of the values $m(v_1)$, $m(v_2)$ and $m(v_3)$ are equal to* 1.

Proof: We consider the principal submatrix of M induced by v_1, v_2, v_3, which is

$$M[K_3] = \begin{pmatrix} m(v_1) & 1 & 1 \\ 1 & m(v_2) & 1 \\ 1 & 1 & m(v_3) \end{pmatrix}.$$

Note that $m(v_1)$, $m(v_2)$ and $m(v_3)$ are positive integers. Since $\det(M[K_3]) = 0$, we have $m(v_1)m(v_2)m(v_3) + 2 = m(v_1) + m(v_2) + m(v_3)$. It leads to

$$(m(v_1)m(v_3) - 1)m(v_2) = m(v_1) + m(v_3) - 2. \tag{2}$$

If $m(v_1)m(v_3) = 1$, then $m(v_1) = m(v_3) = 1$. The result follows.
If $m(v_1)m(v_3) \neq 1$, then we have

$$m(v_2) = \frac{m(v_1) + m(v_3) - 2}{m(v_1)m(v_3) - 1}.$$

Note that $m(v_1)m(v_3) \geq m(v_1) + m(v_3)$ whenever $m(v_1), m(v_3) \geq 2$. Thus at least one of $m(v_1)$ and $m(v_3)$ is equal to 1. Without loss of generality, suppose $m(v_1) = 1$, then

$$m(v_2) = \frac{m(v_3) - 1}{m(v_3) - 1} = 1.$$

Therefore, $m(v_1) = m(v_2) = 1$ and our result follows. □

Recall that $J(s,t) = sK_1\nabla(K_t \cup K_1)$ (see Figure 2) for any positive integers s and t. Now we give the following result.

Lemma 3.8: *If $G \in \mathcal{G}_Q(n, n-2)$ with $n \geq 4$, then either G contains no induced $J(1,2)$ or $G = J(3,5)$.*

Proof: It suffices to prove that $G = J(3,5)$ when G contains an induced $J(1,2)$. Assume that $J(1,2)$ is an induced subgraph of G. Suppose that $J(s,t)$ (see Figure 2) is the maximal induced subgraph of G containing $J(1,2)$ and the vertices of $J(s,t)$ are labelled as in Figure 2, where $U = \{u_1, u_2, \ldots, u_s\}$ and v_0 induce $K_{1,s}$, $V = \{v_1, v_2, \ldots, v_t\}$ induces K_t. Let $M = \mathcal{D}^Q(G) - \partial_n^Q I$ and $m(v) = Tr(v) - \partial_n^Q$ for $v \in V(G)$.

We prove that $m(v_i) = 1$ for $i = 1, 2, \ldots, t$. Note that any three vertices in V form an induced K_3. By Lemma 3.7, at most one vertex in V satisfying $m(v) \neq 1$. If there exists $1 \leq k \leq t$ such that $m(v_k) \neq 1$, say $k = 1$, then $m(v_i) = 1$ for every $2 \leq i \leq t$. On the one hand, since $v_2v_1u_1$ is an induced K_3 and $m(v_1) \neq 1$, by Lemma 3.7, we have $m(u_1) = 1$. On the other hand, since $v_2u_1v_0$ is an induced P_3 and $m(u_1) = 1 \neq 2$, by Remark 2 (i), we have $m(v_2) = 2$, contrary to $m(v_2) = 1$.

Note that $v_1u_jv_0$ is an induced P_3 for each $1 \leq j \leq s$ and $m(v_1) = 1$. By Remark 2 (iii), we have $m(u_j) = 2$ for $j = 1, 2, \ldots, s$ and $m(v_0) = 5$.

Now we prove that $G = J(s,t)$. Otherwise, there exists $x \in V(G) \setminus V(J(s,t))$ such that $N(x) \cap V(J(s,t)) \neq \emptyset$. We consider the following two cases.

Case 1. $x \sim v_0$;

First, we prove that $x \nsim u_i$ for $i = 1, 2, \ldots, s$. Otherwise, there exists $1 \leq k \leq s$ such that $u_k \sim x$, say $k = 1$ (see Figure 2(1)). Then u_1xv_0 is an induced K_3. By Lemma 3.7, at least two of $m(u_1)$, $m(x)$ and $m(v_0)$ are equal to 1, which is impossible because $m(u_1) = 2$ and $m(v_0) = 5$.

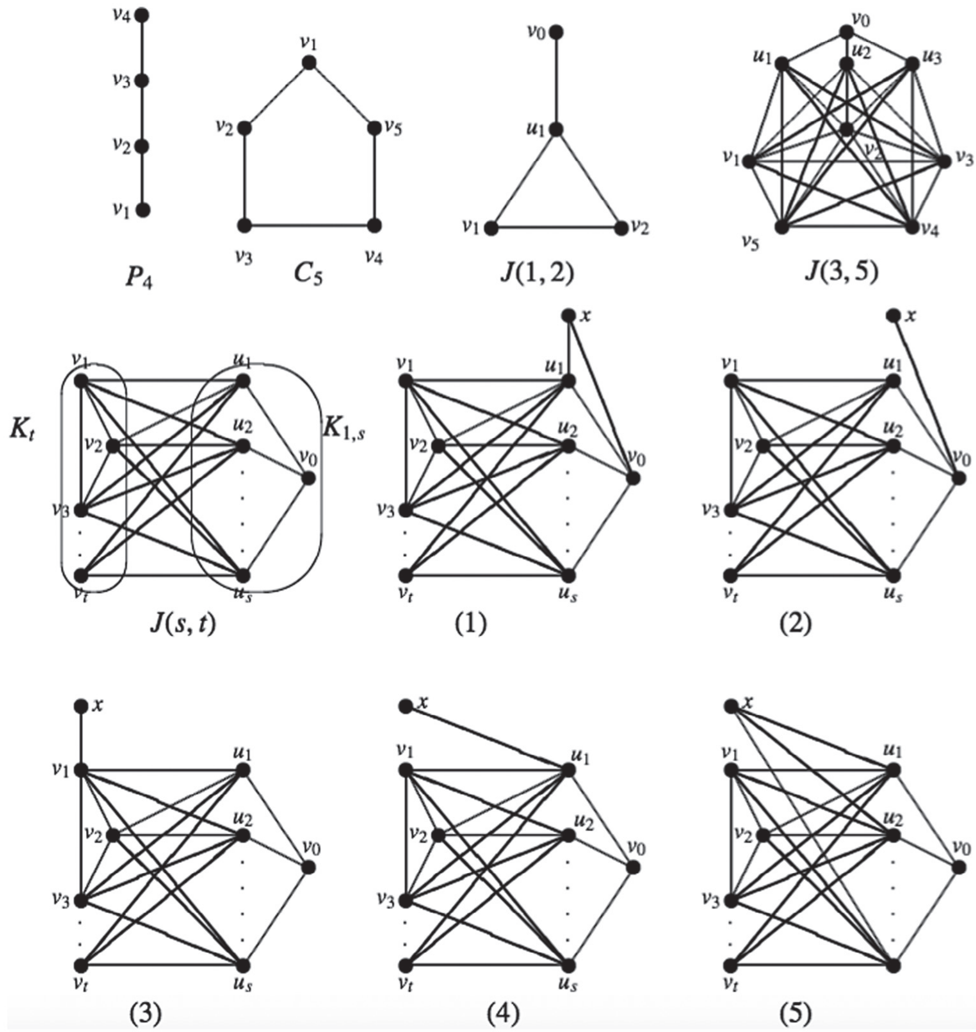

Figure 2. The graphs we need.

Next, we prove that $x \sim v_i$ for $i = 1, 2, \ldots, t$. Otherwise, there exists $1 \leq k \leq t$ such that $v_k \not\sim x$, say $k = 1$ (see Figure 2(2)). Then $v_1u_1v_0x$ is an induced P_4, contradicts Lemma 3.6.

By the arguments above, $V(J(s,t)) \cup \{x\}$ induces the subgraph $J(s+1,t)$, which is impossible because $J(s,t)$ is assumed to be the maximal induced subgraph containing $J(1,2)$.

Case 2. $x \not\sim v_0$.

First, we prove that $N(x) \cap U \neq \emptyset$. Otherwise, $N(x) \cap U = \emptyset$, and thus $N(x) \cap V \neq \emptyset$ because $N(x) \cap V(J(s,t)) \neq \emptyset$. Without loss of generality, suppose that $v_1 \sim x$ (see Figure 2(3)). Then $xv_1u_1v_0$ is an induced P_4, contrary to Lemma 3.6.

Next, we prove that $x \sim u_i$ for $i = 1, 2, \ldots, s$. Since $N(x) \cap U \neq \emptyset$, we may assume that $u_1 \sim x$. If there exists $2 \leq k \leq s$ such that $u_k \not\sim x$, say $k = 2$ (see Figure 2(4)), then $xu_1v_0u_2$ is an induced P_4, also contrary to Lemma 3.6.

At last, we prove that $x \sim v_i$ for $i = 1, 2, \ldots, t$. Since xu_1v_0 is an induced P_3 and $m(v_0) = 5$, by Remark 2 (iii), we have that $m(x) = 1$. If there exists $1 \leq k \leq t$ such that $v_k \not\sim x$, say $k = 1$(see Figure 2(5)), then v_1u_1x is an induced P_3. Since $m(v_1) = 1$, by Remark 2 (iii), we have that $m(x) = 5$, a contradiction.

By the arguments above, $V(J(a, b)) \cup \{x\}$ induces the subgraph $J(s, t + 1)$, which is impossible because $J(s, t)$ is assumed to be the maximal induced subgraph containing $J(1, 2)$.

On the one hand, we have proved that $G = J(s, t)$ and so $Tr(v_1) = s + t + 1$, $Tr(u_1) = 2s+t-1$ and $Tr(v_0) = s+2t$. Note that $m(v_1) = Tr(v_1)-\partial_n^Q = 1$, $m(u_1) = Tr(u_1)-\partial_n^Q = 2$ and $m(v_0) = Tr(v_0) - \partial_n^Q = 5$. Thus we have

$$(s + t + 1) - 1 = (2s + t - 1) - 2 = (s + 2t) - 5 = \partial_n^Q,$$

and thus $s = 3$, $t = 5$ and $n = s + t + 1 = 9$, i.e. $G = J(3, 5)$. On the other hand, by simple computation, we obtain $\mathrm{Spec}_Q(J(3, 5)) = \left\{[20.12]^1, [11.88]^1, [8]^7\right\}$, and so $J(3, 5) \in \mathcal{G}_Q(9, 7)$.

This completes the proof. □

Using the forbidden subgraphs above, we get the following result.

Theorem 3.3: *If $G \in \mathcal{G}_Q(n, n - 2)$ with $n \geq 4$, then either G is a complete multipartite graph or $G = J(3, 5)$.*

***Proof*:** It suffices that G is a complete multipartite graph when $G \neq J(3, 5)$. By Lemma 3.6, G contains no induced P_4 or C_5. Therefore, we have $d(G) = 2$. By Lemma 3.8, G contains no induced $J(1, 2)$ when $G \neq J(3, 5)$. It follows that G is complete multipartite from Lemma 3.3. □

Now we see that in order to determine the graph set $\mathcal{G}_Q(n, n - 2)$, it suffices to find out the graphs in $\mathcal{G}_Q(n, n - 2)$ among all complete multipartite graphs on n vertices.

Lemma 3.9: *Let G be a complete multipartite graph on $n \geq 3$ vertices. Then $G \in \mathcal{G}_Q(n, n-2)$ if and only if $G = K_{\frac{n}{2},\frac{n}{2}}$ or $K_{n-2}\nabla 2K_1$, where*

$$\begin{cases} \mathrm{Spec}_Q(K_{\frac{n}{2},\frac{n}{2}}) = \left\{[3n-4]^1, [2n-4]^1, \left[\frac{3}{2}n - 4\right]^{n-2}\right\} \\ \mathrm{Spec}_Q(K_{n-2}\nabla 2K_1) = \left\{\left[\frac{3n + \sqrt{n^2 - 4n + 20}}{2} - 1\right]^1,\right. \\ \left.\left[\frac{3n - \sqrt{n^2 - 4n + 20}}{2} - 1\right]^1, [n-2]^{n-2}\right\}. \end{cases}$$

***Proof*:** Let $G = K_{n_1,n_2,\ldots,n_s}$ be a complete s-partite graph, where $1 \leq n_1 \leq \cdots \leq n_s$, $n_1 + n_2 + \cdots + n_s = n$ and, we may further assume that $n_s \geq 2$ since otherwise G will be a complete graph.

Suppose that $s = 2$, then $G = K_{a,b}$. The characteristic polynomial of $\mathcal{D}^Q(K_{a,b})$ was given by Aouchiche and Hansen [2], which is

$$P_Q(x) = \left(x - \frac{5n-8+\sqrt{9(a-b)^2+4ab}}{2}\right)\left(x - \frac{5n-8-\sqrt{9(a-b)^2+4ab}}{2}\right) \times (x-2n+b+4)^{a-1}(x-2n+a+4)^{b-1}.$$

Thus $G \in \mathcal{G}_Q(n, n-2)$ if and only if $a = b$, that is, $G = K_{\frac{n}{2},\frac{n}{2}}$. The distance signless Laplacian spectrum of $K_{\frac{n}{2},\frac{n}{2}}$ is given by

$$\mathrm{Spec}_Q(K_{\frac{n}{2},\frac{n}{2}}) = \left\{[3n-4]^1, [2n-4]^1, \left[\frac{3}{2}n-4\right]^{n-2}\right\}.$$

Suppose that $s \geq 3$, we first prove that $G = K_{1,1,\ldots,2} = K_{n-2}\nabla 2K_1$ if $G \in \mathcal{G}_Q(n, n-2)$. Let $V_1, V_2, \ldots, V_s$ be the color sets of G with $|V_i| = n_i$ for $i = 1, 2, \ldots, s$. Recall that $M = \mathcal{D}^Q(G) - \partial_n^Q I$ and $m(v) = Tr(v) - \partial_n^Q$. Note that $m(u) = m(v)$ for any u, v in the same color set because $Tr(u) = Tr(v)$. Let $v_1, v_2 \in V_s$ and $x \in V_i$, $y \in V_j$ where $1 \leq i, j \leq s-1$ and $i \neq j$. On the one hand, since v_1xv_2 is an induced P_3, by Lemma 3.5, we have $m(v_1) = m(v_2) = 2$ because $m(v_1) = m(v_2)$. On the other hand, since xyv_1 is an induced K_3, by Lemma 3.7, at least two of $m(x), m(y), m(v_1)$ are equal to 1, and thus $m(x) = m(y) = 1$ because $m(v_1) = 2 \neq 1$. If V_i contains another vertex $x' \neq x$, then $m(x) = m(x') = 1$. However, since xv_1x' is an induced P_3 and $m(x) = 1$, by Remark 2 (iii), we have $m(x') = 5$, a contradiction. Thus, V_i contains only one vertex. By the arbitrariness of i, we have $G = K_{1,1,\ldots,1,n_s}$. Then $Tr(v_1) = n + n_s - 2$ and $Tr(x) = n - 1$. Note that $m(v_1) = Tr(v_1) - \partial_n^Q = 2$ and $m(x) = Tr(x) - \partial_n^Q = 1$. We have

$$(n + n_s - 2) - 2 = (n-1) - 1 = \partial_n^Q.$$

which implies that $n_s = 2$. Thus, $G = K_{1,1,\ldots,1,2} = K_{n-2}\nabla 2K_1$.

Conversely, we show that $K_{n-2}\nabla 2K_1 \in \mathcal{G}_Q(n, n-2)$. Denote by $U = \{u_1, u_2, \ldots, u_{n-2}\}$ the set of vertices of degree $n-1$ in $K_{n-2}\nabla 2K_1$ and $V = \{v_1, v_2\}$ the set of vertices of degree $n-2$. For $i = 2, \ldots, n-2$, we construct the independent vectors x_i and the vector y as follows

$$x_i(v) = \begin{cases} 1, & v = u_1 \\ -1, & v = u_i \\ 0, & \text{otherwise} \end{cases} \quad \text{and } y(v) = \begin{cases} 1, & v = v_1 \\ -1, & v = v_2 \\ 0, & \text{otherwise} \end{cases}.$$

It is easy to verify that x_i and y are eigenvectors of $\mathcal{D}^Q(K_{n-2}\nabla 2K_1)$ with respect to the eigenvalue $n-2$. Now we construct another vector $z \neq 0$ orthogonal to x_i and y by

$$z(v) = \begin{cases} \alpha, & v \in \{u_1, \ldots, u_{n-2}\} \\ \beta, & v \in \{v_1, v_2\} \end{cases}.$$

Assume that $\mathcal{D}^Q(K_{n-2}\nabla 2K_1)z = \lambda z$. We get $\lambda = \frac{3n\pm\sqrt{n^2-4n+20}}{2} - 1 > n-2$, which are two other eigenvalues of $\mathcal{D}^Q(K_{n-2}\nabla 2K_1)$. Therefore, the distance signless Laplacian spectrum of $K_{n-2}\nabla 2K_1$ is

$$\mathrm{Spec}_Q(K_{n-2}\nabla 2K_1)$$
$$=\left\{\left[\frac{3n+\sqrt{n^2-4n+20}}{2}-1\right]^1,\left[\frac{3n-\sqrt{n^2-4n+20}}{2}-1\right]^1,[n-2]^{n-2}\right\}.$$

Thus, $K_{n-2}\nabla 2K_1 \in \mathcal{G}_Q(n, n-2)$. □

Combining Theorem 3.3 and Lemma 3.9, we obtain our main result.

Theorem 3.4: *For an integer $n \geq 3$, we have*

$$\begin{cases} \mathcal{G}_Q(n,n-2) = \left\{K_{\frac{n}{2},\frac{n}{2}}, K_{n-2}\nabla 2K_1 = K_{1,1,\ldots,1,2}\right\}, & \text{if } n \neq 9 \\ \mathcal{G}_Q(n,n-2) = \left\{J(3,5), K_7\nabla 2K_1 = K_{1,1,\ldots,1,2}\right\}, & \text{if } n = 9 \end{cases}$$

***Proof*:** Suppose that $n \geq 3$. Let $G \in \mathcal{G}_Q(n, n-2)$. If $n = 3$, we get that $G = K_{1,2}$ by immediate calculation. If $n \geq 4$, by Theorem 3.3, we get that either $G = J(3,5) \in \mathcal{G}_Q(9,7)$ or G is a complete multipartite graph. By Lemma 3.9, we have that $G = K_{\frac{n}{2},\frac{n}{2}}$ or $G = K_{1,1,\ldots,1,2} = K_{n-2}\nabla 2K_1$ when $G \neq J(3,5)$. It follows our result. □

We end up this paper by the following result, which is immediate by Theorem 3.4 and by comparing the distance signless Laplacian spectra of $K_{\frac{n}{2},\frac{n}{2}}$ and $K_{n-2}\nabla 2K_1$.

Corollary 3.3: *For $n \geq 3$, the graphs $K_{\frac{n}{2},\frac{n}{2}}$ and $K_{n-2}\nabla 2K_1$ are determined by their distance signless Laplacian spectra.*

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