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Spectral properties of the eccentricity matrix of graphs[☆]

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ABSTRACT

The eccentricity matrix $\mathcal{E}(G)$ of a graph G is derived from the distance matrix by keeping for each row and each column only the largest distances and leaving zeros in the remaining ones. The $\mathcal{E}$-eigenvalues of a graph G are those of its eccentricity matrix $\mathcal{E}(G)$. The $\mathcal{E}$-spectrum of G is the multiset of its $\mathcal{E}$-eigenvalues, where the largest one is the $\mathcal{E}$-spectral radius. In this paper, we proceed to study the algebraic properties of the $\mathcal{E}$-spectrum. In particular, we give a condition to connected graphs with cut vertices so that their eccentricity matrices are irreducible. The latter partially answers the problem given in Wang et al. (2018). We determine the lower and upper bounds for the $\mathcal{E}$-spectral radius of graphs, and we identify the corresponding extremal graphs. Finally, we investigate the least $\mathcal{E}$-eigenvalue of graphs, and list the $\mathcal{E}$-eigenvalues of trees with order 8.

1. Introduction

The graphs considered here are simple and connected graphs. Let $G = (V(G), E(G))$ be a graph with order $n = |V(G)|$ and size $m = |E(G)|$. The *distance* $d_G(v, w)$ between two vertices v and w is the minimum length of the paths joining them. Let $D(G) = (d_{uv})$ be the *distance matrix* of G, where $d_{uv} = d_G(u, v)$. The *eccentricity* $\varepsilon_G(u)$ of the vertex $u \in V(G)$ is given by $\varepsilon_G(u) = \max\{d(u, v) \mid v \in V(G)\}$. Subscripts and arguments in the notation will be omitted, when they are clear from the context. The *radius* and *diameter*, denoted by $\mathrm{rad}(G)$ and $\mathrm{diam}(G)$ respectively, are defined as follows:

$$r = \mathrm{rad}(G) = \min\{\varepsilon(u)|u \in V(G)\} \quad \text{and} \quad d = \mathrm{diam}(G) = \max\{\varepsilon(u)|u \in V(G)\}.$$

The *eccentricity matrix* of G, denoted by $\mathcal{E}(G)$, is constructed from the distance matrix $D(G)$, retaining the largest distances in each row and each column, while other elements of the distance matrix are set to zero. To be more precise, the elements of the eccentricity matrix $\mathcal{E}(G) = (\epsilon_{uv})$ of G are defined as follows [26]:

$$\epsilon_{uv} = \begin{cases} d(u, v) & \text{if } d(u, v) = \min\{\varepsilon(u), \varepsilon(u)\}, \\ 0 & \text{otherwise.} \end{cases}$$

which gives an equivalent definition of the $D_{\max}$-matrix, due to Randić [23].

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The eccentricity matrix can be viewed as the opposite of the adjacency matrix. In fact, the adjacency matrix (of a connected graph) is constructed from the distance matrix by keeping for each row and each column only the smallest distances, which correspond to adjacent vertices. From this point of view, these two matrices express two extremes of distance-like matrix. According to Randić [23] the eccentricity matrix might have interesting applications not only in Chemical Graph Theory but also in Graph Theory due to the role of eccentricities in graphs. Very recently, some further spectral properties and the energy of eccentricity matrix are investigated in [22,28].

We now introduce some notation and terminology borrowed from the spectral and chemical graph theory. The (ordinary) graph spectrum is formed by the eigenvalues of the adjacency matrix $A(G)$ [2]. In order to distinguish the eigenvalues of different graph matrices, the eigenvalues of $A(G)$ are said to be *A-eigenvalues*. Correspondingly, the eigenvalues with respect to $D(G)$ and $\mathcal{E}(G)$ are respectively called to the *D-eigenvalues* and the *$\mathcal{E}$-eigenvalues*, and form the *D-spectrum* and the *$\mathcal{E}$-spectrum*, respectively. Since $\mathcal{E}(G)$ is symmetric, the $\mathcal{E}$-eigenvalues are real. Let $\xi_1 > \xi_2 > \cdots > \xi_k$ be all the distinct $\mathcal{E}$-eigenvalues. Therefore, the $\mathcal{E}$-spectrum can be written as

$$\mathrm{Spec}_{\mathcal{E}}(G) = \begin{Bmatrix} \xi_1 & \xi_2 & \cdots & \xi_k \\ m_1 & m_2 & \cdots & m_k \end{Bmatrix},$$

where m_i is the multiplicity of the eigenvalue ξ_i $(1 \le i \le k)$ and ξ_1 is called the *$\mathcal{E}$-spectral radius* of G which is denoted by $\xi(G)$. The spectral radius of a graph matrix is possibly the most investigated eigenvalue, since it is investigated in numerous papers (see, e.g., [15,20,25,29]), there is a survey paper [4] and a monograph [24] collecting many important results.

One of the most important facts is that the adjacency and distance matrix of connected graphs is irreducible, but it may unfit for the eccentricity matrix. In fact, let K_n and $K_{a,b}$ denote, respectively, the complete graph and complete bipartite graph, then $\mathcal{E}(K_n) = J_n - I_n$ is irreducible, while, $\mathcal{E}(K_{a,b}) = \begin{pmatrix} 2(J_a - I_a) & O \\ O & 2(J_b - I_b) \end{pmatrix}$ is reducible, where I_n and J_n are the identity matrix and all-one matrix, respectively. The problem of characterizing the connected graphs whose eccentricity matrix is irreducible naturally arises. Thus, the authors [26] put forward the following question:

For which connected graphs the eccentricity matrix is either reducible or irreducible?

Very recently, Wang et al. [26] showed that the eccentricity matrix of trees is irreducible, and further investigated the relations between the eigenvalues of the adjacency and eccentricity matrices. They also gave some applications of this new matrix in terms of molecular descriptors. For others details on this new matrix and the various Randić-type descriptors, we refer the readers to [9,21,23]. In this paper, we proceed to investigate the $\mathcal{E}$-spectra of graphs.

The paper is organized as follows. In Section 2, we give a sufficient condition for graphs with cut vertices to get a irreducible eccentricity matrix. In Section 3, we determine the upper bound of the least $\mathcal{E}$-eigenvalues. In Section 4, we determine the upper and lower bounds of $\mathcal{E}$-spectral radius of graphs, and characterize the corresponding extreme graphs. In Section 5, we give some concluding remarks and we propose further directions of research for this graph matrix. In the appendix we depict the trees of order 8 and we list the corresponding $\mathcal{E}$-spectra.

2. Irreducibility of eccentricity matrix

A matrix M is said to be irreducible if it is not permutationally similar to an upper triangular block matrix. In other words M is reducible if there exists a permutation matrix P such that

$$M = P^T \begin{pmatrix} A & B \\ O & C \end{pmatrix} P,$$

where A and C are square blocks.

To determine whether a matrix is irreducible, it is useful to consider its corresponding matrix graph. For a symmetric matrix M of order n, its *matrix graph* G^M is defined as follows. The vertices of G^M are $1, \ldots, n$, and distinct vertices i, j are adjacent if and only if $M_{ij} \neq 0$. It is well-known that M is irreducible if and only if G^M is connected (see [17], Theorem 6.2.24). Using this statement, we give the following example.

Example 1. Let C_n be the cycle with vertex set $\{v_0, \ldots, v_{n-1}\}$ such that $v_i \sim v_{i+1}$ for $0 \le i \le n-1 (\mathrm{mod}\ n)$. Denote by $\mathcal{E}(C_n) = (\epsilon_{ij})_{n\times n}$ the eccentricity matrix of C_n and G^E the matrix graph of $\mathcal{E}(C_n)$. Note that $\varepsilon(v) = \lfloor n/2 \rfloor$ for any $v \in V(C_n)$ and $d(v_i, v_j) = \min\{j-i, n+i-j\}$ for any $v_i, v_j \in V(C_n)$ and $i < j$. We have $d(v_i, v_j) = \min\{\varepsilon(v_i), \varepsilon(v_j)\} = \lfloor n/2 \rfloor$ if and only if $j - i = \lfloor n/2 \rfloor$ or $\lceil n/2 \rceil$ if and only if $v_i \sim v_j$ in G^E. If $n = 2k$, then $v_i \sim v_{i+k} (\mathrm{mod}\ n)$ in G^E for $0 \le i \le n$. For $0 \le i \le k$, since $i+k+k = i (\mathrm{mod}\ n)$, the vertices $\{v_i, v_{i+k}\}$ induces a K_2 which is a connected component of G^E. Therefore, we have $G^E = kK_2$ and thus $\mathcal{E}(C_{2k})$ is reducible. If $n = 2k+1$, then $v_0 \sim v_k \sim v_{2k} \sim \cdots \sim v_{ik} \sim v_{(i+1)k} \sim \cdots$ in G^E. Since $\gcd(k, 2k+1) = 1$, we have $ik = jk (\mathrm{mod}\ 2k+1)$ if and only if $j = i (\mathrm{mod}\ 2k+1)$. It means that $v_0, v_k, \ldots, v_{(2k-1)k}, v_{2k^2}$ are all distinct. Therefore, $G^E = v_0 v_k v_{2k} \cdots v_{2k^2}$ is a cycle of order $2k+1$ and thus $\mathcal{E}(C_{2k+1})$ is irreducible.

From this example, we find that it seems not easy to determine whether the eccentricity matrix of a graph is irreducible or not. In what follows we give a class of graphs whose eccentricity matrices are irreducible.

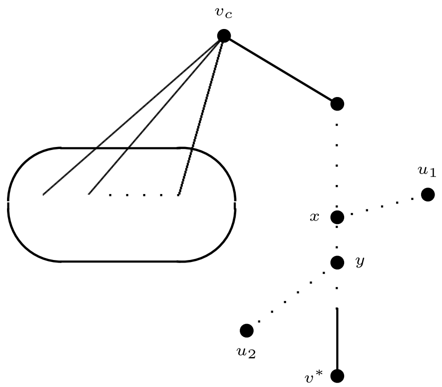

Fig. 1. The graph in the proof of Theorem 2.1.

Theorem 2.1. *Let G be a connected graph with order n and a cut vertex $v_c \in V(G)$. If $G - v_c$ contains a component T which is a tree such that $\varepsilon_G(v_c) = d(v_c, v)$ for some $v \in V(T)$, then $\mathcal{E}(G)$ is irreducible.*

Proof. Let A' be the matrix obtained from $\mathcal{E}(G)$ by setting to 1 each nonzero entry of $\mathcal{E}(G)$. Let G' such that $A' = A(G')$. If so, A' is irreducible if and only if G' is a connected graph. Evidently, if A' is irreducible, then we also obtain that $\mathcal{E}(G)$ is irreducible, as well.

Let $T^* = \{v \in T : d(v_c, v) = \varepsilon(v_c)\}$. In view of the assumptions, we have $T \neq \emptyset$, and let $v^* \in T^*$. Obviously, v_c is adjacent (in G') to each vertex of T^*.

Consider now any vertex $u \in G \setminus T$, and let v_u be a vertex at maximum distance from u. We have the following chain of inequalities:

$$\varepsilon(u) = d(u, v_u) \leq d(u, v_c) + d(v_c, v_u) \leq d(u, v_c) + d(v_c, v^*) = d(u, v^*),$$

where the last equality is due by the assumption of v_c being a cut-vertex. By the above chain, we get that u is adjacent to v^* in G'.

Now we focus our attention to the vertices in T. If $u \in T$ is a vertex whose most distant vertex v_u is in $G \setminus T$, then u is adjacent (in G') to some vertex of $G \setminus T$, and we are done due to the previous case. Also, if $u \in T^*$, we are also done, since u is adjacent to v_c. Hence, it remains to consider the vertices $u \in T \setminus T^*$ whose most distant vertex v_u is within T. We have the following subcases:

Case 1. The path between u and v_u (the (u, v_u)-path) does not intersect the path between v_c and v^*. In the latter case, let x be the vertex in the (u, v_u)-path closest to the vertex y in the (v_c, v^*)-path. Since $\varepsilon(v_c) = d(v_c, v^*)$, it is $d(y, v_u) \leq d(y, v^*)$. Therefore, we get

$$d(u, v_u) = d(u, x) + d(x, v_u) < d(u, x) + d(y, v_u) < d(u, x) + d(x, y) + d(y, v^*) = d(u, v^*),$$

that is a contradiction to v_u being at maximal distance from u. Hence, the (u, v_u)-path must encounter the (v_c, v^*)-path.

Case 2. u takes the role of x, and v_u is either u_1 or u_2 (cf. Fig. 1). If so, we have the following chain of inequalities:

$$d(v_c, v_u) = d(v_c, u) + d(u, v_u) \leq d(v_c, u) + d(u, v^*) = d(v_c, v^*).$$

The latter means that $d(u, v_u) = d(u, v^*)$, so u is adjacent to v^* and we are done.

Case 3. u takes the role of u_1 and v_u takes the role of u_2, x and y might be the same vertex (cf. Fig. 1). If so, note that because of $\varepsilon(v_c) = d(v_c, v^*)$, it is $d(x, v_u) \leq d(x, v^*)$, therefore

$$d(u, v_u) = d(u, x) + d(x, v_u) \leq d(u, x) + d(x, v^*) = d(u, v^*),$$

and again u is adjacent to v^* in G'.

Case 4. u takes any other position and v_u is some (pendant) vertex of T. If so, u is adjacent to $v_u = w_1$ in G'. Let us consider w_1 instead. If the most distant vertex from w_1, say v_{w_1}, lies in $G \setminus T$, or in T^*, or it falls in Case 3, we are done since u is adjacent to w_1, and w_1 is adjacent to v^* or v_c. If $v_{w_1} = w_2 \in T$ does not fall in any of the previous situations, it means that w_1 is positioned as u_2 and w_2 is positioned as u_1 (according to Fig. 1). In such situation, we consider $v_{w_2} = w_3$ and check its position. Note that if $w_3 = w_1$, then we are also done, as we get the configuration described in Case 3.

Eventually, we can reuse the latter routine up to some w_n which, due to T being finite, falls in Case 3 (or it is in T^* or $G \setminus T$), and then u is joined to v^* or to v_c by the path (in G') of vertices w_i.

Thus, for all $u \in G'$, $(u, v_c) \in E(G')$, and then the proof is complete. □

The following two corollaries immediately follows from Theorem 2.1.

Corollary 2.2 ([26]). *Let T be a tree. Then $\mathcal{E}(T)$ is irreducible.*

Let G_1 and G_2 be two disjoint rooted graphs, whose roots are $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$, respectively. The graph $G_1\nabla_{v_1v_2}G_2$ is obtained by identifying v_1 and v_2.

Corollary 2.3. *Let H be a connected graph and $v \in V(H)$. If T is tree with a pendant vertex $v^* \in V(T)$ such that $\varepsilon_H(v) \le \varepsilon_T(v^*)$, then the eccentricity matrix of the graph $G = H\nabla_{vv^*}T$, $\mathcal{E}(G)$, is irreducible.*

3. The least $\mathcal{E}$-eigenvalue of graphs

The least eigenvalue of graphs has been often studied in the literature. For example, several scholars have paid attention to distance-regular graphs with extremal least A-eigenvalue (cf. [1,12,19]), and the graphs with least A-eigenvalue not less than -2 which are the generalized line graphs and the graphs represented in root system E_8 (cf. [3,5] for example).

We here consider the least $\mathcal{E}$-eigenvalue of graphs, which involves the so called *antipodal graphs*. Let G be a connected graph with $u \in V(G)$. The set of vertices at distance k from u is denoted by $G_k(u)$ $(0 \le k \le \varepsilon(u))$. A graph G of diameter d is called *antipodal* if, for any given vertex $u \in V(G)$, the set $\{u\} \cup G_d(u)$ consists of vertices which are mutually at distance d. In other words, there exists a partition of the vertex set into classes (called the *fibres* of G) with the property that two distinct vertices are in the same class iff they are at distance d (see, for instance, Godsil [13]). If all the fibres have the same cardinality, say r, we say that G is an r-antipodal graph. Obviously, the complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$ is an antipodal graphs. Moreover, the complete graph K_n is n-antipodal. The regular multipartite graph $K_{a,a,\ldots,a}$, the even cycle C_{2t} and the cube are 2-antipodal graphs. It is easy to see that the $\mathcal{E}$-matrix of r-antipodal graph G with diameter d $(r \ge 2)$ is as follows

$$\mathcal{E}(G) = \begin{pmatrix} d(J_r - I_r) & & & \\ & d(J_r - I_r) & & \\ & & \ddots & \\ & & & d(J_r - I_r) \end{pmatrix} \tag{1}$$

whose least $\mathcal{E}$-eigenvalue is $-d$.

Theorem 3.1. *Let G be a connected graph with order n and diameter d, and let $\xi_n(G)$ be the least $\mathcal{E}$-eigenvalue of G. Then $\xi_n(G) \le -d$, with equality only if G is an r-antipodal graph with diameter d $(r \ge 2)$. In particular,*

(i) *$\xi_n(G) = -1$ if and only if G is the complete graph K_n $(n \ge 2)$.*
(ii) *$\xi_n(G) = -2$ if and only if G is the complete multipartite graph $K_{n_1,n_2,\ldots,n_k}$ $(n \ge 4, k \ge 2)$.*

Proof. Let P be the diametrical path of G with diametrical vertices u and v. Then $d_P(u, v) = d$, and the submatrix of $\mathcal{E}(G)$ index by u, v of is $B = \begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix}$ with the least eigenvalue $\lambda(B) = -d$. Hence, $\xi_n(G) \le \lambda(B) = -d$. If G is r-antipodal $(r \ge 2)$, by (1) we get $\xi_n(G) = -d$.

Clearly, (i) holds. We next show (ii). By $\xi_n(G) = -2$ we get $\mathrm{diam}(G) = 2$. Clearly $n \ge 4$, and we get two cases.

Case 1. The maximum degree $\Delta(G) \le n - 2$. Then$\mathcal{E}(G) = 2A(\overline{G})$. Hence, the least A-eigenvalue of the complementary graph $\overline{G}$ is -1, and so $\overline{G}$ is the disjoint union of some complete graphs K_{n_i} $(i = 1, 2, \ldots, n)$. Thereby, G is the complete multipartite graph $K_{n_1,n_2,\ldots,n_k}$.

Case 2. $\Delta(G) = n - 1$. If G is a tree, then $G \cong K_{1,n-1}$ with $\xi_n(G) = -2$. Otherwise, by $\Delta(G) = n - 1$ and $\mathrm{diam}(G) = 2$ we get that G contains G_1 and G_2 as its subgraphs, where G_1 is the triangle with a pendant edge and $G_2 = K_{1,1,2}$. For the former, the submatrix indexed by $V(G_1)$ is always

$$\mathcal{E}(G_1) = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 2 & 2 & 1 & 0 \end{pmatrix}.$$

Therefore, $\xi_n(G) \le \xi_n(G_1) \approx -2.86489$, a contradiction. For the latter, if $n = 4$ then $G = G_2$ and thus $\xi_n(G) = -2$, if $n \ge 5$ then G_1 is a subgraph of G, a contradiction, as well. □

Graphs with the same $\mathcal{E}$-spectrum are called *$\mathcal{E}$-cospectral graphs* graphs. A graph G is said to be *determined by its M-spectrum* if there is no other non-isomorphic graph with the same spectrum. This problem was first considered in Chemistry [14], and we refer the readers to [7,8] for details on this topic. We next consider the spectral determination.

Corollary 3.2. *The graphs K_n and $K_{a_1,a_2,\ldots,a_k}$ are determined by their $\mathcal{E}$-spectra.*

Proof. Let H be a graph $\mathcal{E}$-cospectral to $G = K_{a_1,a_2,\ldots,a_k}$. So, $n(H) = n(G) = n$. By Theorem 3.1(ii) we get $H = K_{b_1,b_2,\ldots,b_s}$. Easily to get the $\mathcal{E}$-eigenvalues of G are $2(a_i - 1)$ $(1 \le i \le k)$ and -2 with multiplicity $n - k$. Hence, $k = s$ and $a_i = b_i$ $(1 \le i \le k)$, and thus H and G are isomorphic. □

Remark 3.3. About Theorem 3.1 and Corollary 3.2,

(i) The complete multipartite graph $K_{a_1,a_2,\ldots,a_n}$ is determined by the $\mathcal{D}$-spectrum [18]; while it remains open for the A-spectrum.
(ii) By Theorem 3.1(ii) we get $\xi(T) \le -2$ with equality if and only if $T = K_{1,n-1}$ when $G = T$ is a tree. This solves a conjecture posed by Wang et al. ([26], Conjecture 2), which was proposed by observing the list of $\mathcal{E}$-eigenvalues of trees with order at most 7.

4. The bounds of $\mathcal{E}$-spectral radius of graphs

In this section, we investigate the lower and upper bounds of $\mathcal{E}$-spectral radius of graphs and characterize the corresponding extremal graphs. We next point out several basic properties about the $\mathcal{E}$-spectral radius of graphs.

Let $B = (B_{ij})$ be an $m \times n$ matrix. Then $s_i(B)$ will denote the ith row sum of B, i.e., $s_i(B) = \sum_{j=1}^{n} B_{ij}$, where $1 \le i \le m$. Let $\rho(B)$ be the spectral radius of B.

Lemma 4.1. *Let B be a $n \times n$ matrix.*

(i) *[11] If B is a real symmetric matrix, and if λ is an eigenvalue of B with an eigenvector x whose entries are nonnegative. Then*

$$\min_{1\le i\le n} s_i(B) \le \lambda \le \max_{1\le i\le n} s_i(B).$$

(ii) *[17] If B is nonnegative, then there exists a nonnegative eigenvector $\mathbf{x}$ such that $B\mathbf{x} = \rho(B)\mathbf{x}$.*

Observe that the eccentricity matrix of a connected nontrivial graph is a real symmetric nonnegative matrix whose rows each have a positive element. The following result follows from the above lemma and the algebraic properties of $\mathcal{E}(G)$.

Lemma 4.2. *Let G be a connected nontrivial graph with ξ-spectral radius $\xi(G)$.*

(i) *Then $\xi(G) > 0$ and it has an eigenvector with nonnegative entries.*
(ii) $\sum_{i=1}^{n} \xi_i = 0$ *and* $S = \sum_{i=1}^{n} \xi_i^2 = 2\sum_{1\le i<j\le n} \mathcal{E}_{ij}^2(G)$.

As mentioned in the introduction, the eccentricity matrices of connected graphs are not irreducible in general (but they are irreducible for trees [26]). Hence, the theory of the nonnegative irreducible matrix cannot be directly applied to eccentricity matrices. In order to obtain the main result in this section, we need a result due to Hoffman. Let G be a graph with k distinct A-eigenvalues $\mu_1 > \mu_2 > \cdots > \mu_k$. Hoffman [16] shown that G is a connected regular graph with order n if and only if

$$n\prod_{i=2}^{k}(A - \mu_i I_n) = \prod_{i=2}^{k}(\mu_1 - \mu_i)J_n$$

holds.

Dress and Stevanović [10] generalized the above result to real symmetric matrices in the following way. Let B be a real symmetric $n\times n$ matrix over real field $\mathbb{R}$, and $\{\beta_1, \beta_2, \ldots, \beta_t\}$ be any family of elements from $\mathbb{R}$. Let $s(B) = \{\mu | \dim V_\mu > 0\}$ be the set of its distinct eigenvalues, and $V_\mu = \{\alpha_\mu \mid M\alpha_\mu = \mu\alpha_\mu\}$ be the eigenspace w.r.t. the eigenvalue μ. If $\alpha_{\mu_1}, \alpha_{\mu_2}, \ldots, \alpha_{\mu_{r_\mu}}$ is an orthonormal basis of V_μ for every eigenvalue μ in $s(B)$, then they [10] obtained

$$\prod_{i=1}^{t}(B - \beta_i I) = \sum_{\mu\in s(B)\setminus\{\beta_1,\ldots,\beta_t\}} \prod_{i=1}^{t}(\mu - \beta_i)\sum_{i=1}^{r_\mu} \alpha_{\mu_i}\alpha_{\mu_i}^T. \tag{2}$$

Since we are interested in the largest eigenvalue, we give a variant of the Hoffman-type identity in terms of Hermitian matrices with simple spectral radius.

Let $\mathbf{y}^* = \bar{y}^T$ be the transpose of complex conjugates of the vector $\mathbf{y}$.

Theorem 4.3. *Let H be a Hermitian matrix of order n with simple spectral radius. Then H has exactly $k \ge 2$ distinct eigenvalues if and only if there exists k real numbers $\lambda_1, \lambda_2, \ldots, \lambda_k$ with $\lambda_1 > \lambda_2 > \cdots > \lambda_k$ such that*

$$\prod_{j=2}^{k}(H - \lambda_j I) = \prod_{j=2}^{k}(\lambda_1 - \lambda_j)\mathbf{y}\mathbf{y}^* \quad \text{and} \quad H\mathbf{y} = \lambda_1\mathbf{y}. \tag{3}$$

Moreover, $\lambda_1 > \lambda_2 > \cdots > \lambda_k$ are the distinct eigenvalues of H.

Proof. Note that the necessity can be derived from (2). Here, we provide a more elementary proof.

Let $\lambda_1 > \lambda_2 > \cdots > \lambda_k$ be the k distinct eigenvalues of H with λ_1 being simple. Let $f(x) = \prod_{i=2}^{k}(x-\lambda_i)$. Then the eigenvalues of $f(H)$ are $f(\lambda_1)$ with multiplicity one and 0 with multiplicity $n-1$. Hence, the rank of $f(H)$ is one, and consequently there exist two non-zero n-vectors $\mathbf{x}, \mathbf{y}$ such that

$$f(H) = \mathbf{x}\mathbf{y}^* \quad \text{and} \quad f(H)\mathbf{x} = (\mathbf{y}^*\mathbf{x})\mathbf{x}. \tag{4}$$

The second one of (4) indicates that $\mathbf{y}^*\mathbf{x}$ is just the only one non-zero eigenvalue of $f(H)$. Actually, $\mathbf{y}^*\mathbf{x} = f(\lambda_1)$. Due to the first one of (4), we get $\mathbf{y}^*f(H) = f(\lambda_1)\mathbf{y}^*$ which leads to

$$f(H)\mathbf{y} = f(H)^*\mathbf{y} = f(\lambda_1)\mathbf{y}.$$

Thereby, both $\mathbf{x}$ and $\mathbf{y}$ are the eigenvectors of $f(H)$ associated with eigenvalue $f(\lambda_1)$. Since $f(H)$ is diagonalizable, then geometric multiplicity of $f(\lambda_1)$ is one. Hence, there exists a non-zero number $b \in \mathbb{R}$ such that $\mathbf{x} = b\mathbf{y}$. Therefore,

$$f(H) = \prod_{j=2}^{k}(H - \lambda_j I) = b\mathbf{y}\mathbf{y}^*. \tag{5}$$

Note that the minimal polynomial of H is $m(x) = \prod_{i=1}^{k}(x-\lambda_i)$ and $m(H) = \prod_{i=1}^{k}(H-\lambda_i I) = O$. Then, by (5) we obtain

$$b(H-\lambda_1 I)\mathbf{y}\mathbf{y}^* = O, \quad \text{that is,} \quad \|\mathbf{y}\|_2^2(H-\lambda_1 I)\mathbf{y} = \mathbf{0}. \tag{6}$$

Since $\mathbf{y} \neq \mathbf{0}$, then $\|\mathbf{y}\|_2^2 > 0$ which along with (6) forces to $(H-\lambda_1 I)\mathbf{y} = \mathbf{0}$, and consequently

$$H\mathbf{y} = \lambda_1\mathbf{y}.$$

We now show the sufficiency. Since $(H-\lambda_1 I)\mathbf{y} = \mathbf{0}$, we have $\prod_{j=1}^{k}(H-\lambda_j I) = O$. We claim that $p(x) = \prod_{j=1}^{k}(x-\lambda_j)$ is the minimal polynomial of H. Since otherwise, the minimal polynomial $q(x)|p(x)$ where $\deg(q(x)) < \deg(p(x))$. Note that $x-\lambda_1$ must be a factor of $q(x)$ due to λ_1 is assumed to be an eigenvalue. Without loss of generality, assume that $q(x) = \prod_{j=1}^{k'}(x-\lambda_j)$ where $k' < k$. Thus $\prod_{j=1}^{k'}(H-\lambda_j I) = O$. If $k' = 1$ then $H = \lambda_1 I$, which, returns to (3), will produce a contradiction. Hence $k' \geq 2$ and then $(x-\lambda_1)(x-\lambda_2)|q(x)$. As proved in the necessity, from (5) it follows that there exists $c \neq 0$ such that $\prod_{j=2}^{k'}(H-\lambda_j I) = c\mathbf{y}\mathbf{y}^*$. Thus

$$\prod_{j=2}^{k}(\lambda_1-\lambda_j)\prod_{j=2}^{k'}(H-\lambda_j I) = c\prod_{j=2}^{k}(\lambda_1-\lambda_j)\mathbf{y}\mathbf{y}^* = c\prod_{j=2}^{k}(H-\lambda_j I).$$

Set $f(x) = \prod_{j=2}^{k}(\lambda_1-\lambda_j)\prod_{j=2}^{k'}(x-\lambda_j) - c\prod_{j=2}^{k}(x-\lambda_j)$. Then $f(H) = O$, and hence $q(x)|f(x)$. Thus $f(\lambda_1) = f(\lambda_2) = 0$, which gives that

$$\prod_{j=2}^{k'}(\lambda_1-\lambda_j) = c = \frac{\prod_{j=2}^{k}(\lambda_1-\lambda_j)}{\prod_{j=k'+1}^{k}(\lambda_2-\lambda_j)}.$$

Hence $\prod_{j=k'+1}^{k}(\lambda_1-\lambda_j) = \prod_{j=k'+1}^{k}(\lambda_2-\lambda_j)$, which is impossible. Thus $p(x) = \prod_{j=1}^{k}(x-\lambda_j)$ is the minimal polynomial of H. It follows that H has exactly k distinct eigenvalues.

This finishes the proof. □

Note that we can also adopt the spectral decomposition of the matrix H to prove the necessity of Theorem 4.3. Recall, $S = 2\sum_{1\leq i<j\leq n}\mathcal{E}_{ij}^2(G)$ defined in Lemma 4.2.

Theorem 4.4. *Let G be connected graph with order $n \geq 2$. Then*

$$\xi_1 \leq \sqrt{\frac{(n-1)S}{n}},$$

where the equality holds iff $\xi_1 \neq \xi_2 = \xi_3 = \cdots = \xi_n$. Furthermore, if $\xi_1 = n-1$, the upper bound is achieved if and only if G is the complete graph.

Proof. From Lemma 4.2(ii) it follows that $\xi_1 = -\sum_{i=2}^{n}\xi_i$ and $S = \sum_{i=1}^{n}\xi_i^2$. By the Cauchy–Schwarz Inequality we get

$$\xi_1^2 = (\sum_{i=2}^{n}\xi_i)^2 \leq (n-1)\sum_{i=2}^{n}\xi_i^2 = (n-1)(S-\xi_1^2),$$

which leads to $n\xi_1^2 \leq (n-1)S$, and so the inequality follows.

If $\xi_2 = \xi_3 = \cdots = \xi_n$, then the equality holds. On the other hand, if the equality holds, then by the Cauchy–Schwarz Inequality again, we have $\xi_2 = \xi_3 = \cdots = \xi_n$. Assume that $\xi_1 = \xi_2$. Then $\xi_1 = \xi_2 = \cdots = \xi_n$ which together with Lemma 4.2 results in $\xi_1 = \cdots = \xi_n = 0$, contradicting Lemma 4.2(i). Hence, $\xi_1 \neq \xi_2 = \xi_3 = \cdots = \xi_n$.

We now prove the additional claim.

If G is a complete graph of order n, then $\mathcal{E}(G) = J_n - I_n$. Therefore, its $\mathcal{E}$-eigenvalues are $\xi_1 = n - 1$ with eigenvector $\mathbf{y} = \frac{1}{\sqrt{n}}(1, 1, \ldots, 1)^T$ and $\xi_2 = \cdots = \xi_n = -1$, so the equality holds.

Conversely, if $\xi_1 = n - 1$, $\mathbf{y} = \frac{1}{\sqrt{n}}(1, 1, \ldots, 1)^T$ and $\xi_2 = \cdots = \xi_n = -1$, in view of (3) in Theorem 4.3 we get $H = J_n - I_n$, which implies that G is a complete graph.

This completes the proof. □

It is worth mentioning that in [27] the graphs with two distinct $\mathcal{E}$-eigenvalues are studied, and it is proved that the complete graph is the unique one having simple $\mathcal{E}$-spectral radius.

We conclude this section by determining the lower bounds for the $\mathcal{E}$-spectral radius of graphs. The following two definitions are needed.

Definition 4.1. Let G be a connected graph.

(i) For each vertex $v_i \in V(G)$, its $\mathcal{E}$-degree is defined to be $\mathcal{E}(i) = \sum_{j=1}^{n} \mathcal{E}_{ij}(G)$.

(ii) Let the $\mathcal{E}$-degree sequence of G be $\{\mathcal{E}(1), \mathcal{E}(2), \ldots, \mathcal{E}(n)\}$. Then the second $\mathcal{E}$-degree of v_i is defined as $T_{\mathcal{E}}(i) = \sum_{j=1}^{n} \mathcal{E}_{ij}\mathcal{E}(j)$.

Definition 4.2. Let G be a connected graph with $\mathcal{E}$-degree sequence $\{\mathcal{E}(1), \mathcal{E}(2), \ldots, \mathcal{E}(n)\}$ and the second $\mathcal{E}$-degree sequence $\{T_{\mathcal{E}}(1), T_{\mathcal{E}}(2), \ldots, T_{\mathcal{E}}(n)\}$. Then

(i) G is $\mathcal{E}$-regular if $\mathcal{E}(i) = k$ for all i.

(ii) G is pseudo $\mathcal{E}$-regular if $\frac{T_{\mathcal{E}}(i)}{\mathcal{E}(i)} = k$ for all i.

Theorem 4.5. *Let G be connected graph with $\mathcal{E}$-degree sequence $\{\mathcal{E}(1), \mathcal{E}(2), \ldots, \mathcal{E}(n)\}$ and second $\mathcal{E}$-degree sequence $\{T_{\mathcal{E}}(1), T_{\mathcal{E}}(2), \ldots, T_{\mathcal{E}}(n)\}$. Then*

(i) $\xi_1 \geq \frac{1}{n}\sum_{i=1}^{n} \mathcal{E}(i)$, *where the equality holds if and only if G is $\mathcal{E}$-regular.*

(ii) $\xi_1 \geq \sqrt{\frac{T_{\mathcal{E}}^2(1)+T_{\mathcal{E}}^2(2)+\cdots+T_{\mathcal{E}}^2(n)}{\mathcal{E}^2(1)+\mathcal{E}^2(2)+\cdots+\mathcal{E}^2(n)}}$, *where the quality holds if and only if G is pseudo $\mathcal{E}$-regular.*

Proof. (i) Let $x = \frac{1}{\sqrt{n}}(1, 1, \ldots, 1)$. Applying Rayleigh quotient to $\mathcal{E}(G)$ we obtain

$$\xi_1 \geq \frac{x\mathcal{E}x^T}{xx^T} = \frac{1}{\sqrt{n}}[\mathcal{E}(1), \mathcal{E}(2), \ldots, \mathcal{E}(n)]\frac{1}{\sqrt{n}}(1, 1, \ldots, 1)^T = \frac{1}{n}\sum_{i=1}^{n}\mathcal{E}(i).$$

If G is $\mathcal{E}$-regular, then the sum of each row of $\mathcal{E}(G)$ is a constant k. Thus $\sum_{i=1}^{n} \mathcal{E}(i) = nk$. By Lemma 4.1(i), k is the $\mathcal{E}$-spectral radius. So, $\xi_1 = k = \frac{nk}{n}$, and the equality follows.

Conversely if equality holds, then x is the eigenvector corresponding to ξ_1 and hence $\mathcal{E}(G)x = \xi_1 x$. This then gives $\mathcal{E}(i) = \xi_1$ for all i. Since $\mathcal{E}(i)$ is an integer it follows that G is $\mathcal{E}$-regular.

(ii) Let $\mathcal{E}$ be the eccentricity matrix of G and $X = (x_1, x_2, \ldots, x_n)$ be the unit nonnegative eigenvector of $\mathcal{E}$ corresponding to ξ_1. Set

$$Y = \frac{1}{\sqrt{\sum_{i=1}^{n}\mathcal{E}^2(i)}}(\mathcal{E}(1), \mathcal{E}(2), \ldots, \mathcal{E}(n)).$$

Then Y is unit positive vector. So we get

$$\xi_1 = \xi_1(\mathcal{E}) = \sqrt{\xi_1(\mathcal{E}^2)} = \sqrt{X\mathcal{E}^2X^T} \geq \sqrt{Y\mathcal{E}^2Y^T}.$$

Now

$$Y\mathcal{E} = \frac{1}{\sqrt{\sum_{i=1}^{n}\mathcal{E}^2(i)}}(\mathcal{E}(1), \mathcal{E}(2), \ldots, \mathcal{E}(n))\mathcal{E} = \frac{1}{\sqrt{\sum_{i=1}^{n}\mathcal{E}^2(i)}}(T_{\mathcal{E}}(1), T_{\mathcal{E}}(2), \ldots, T_{\mathcal{E}}(n)).$$

Thereby

$$Y\mathcal{E}^2Y^T = Y\mathcal{E}(Y\mathcal{E})^T = \frac{T_{\mathcal{E}}^2(1) + T_{\mathcal{E}}^2(2) + \cdots + T_{\mathcal{E}}^2(n)}{\mathcal{E}^2(1) + \mathcal{E}^2(2) + \cdots + \mathcal{E}^2(n)},$$

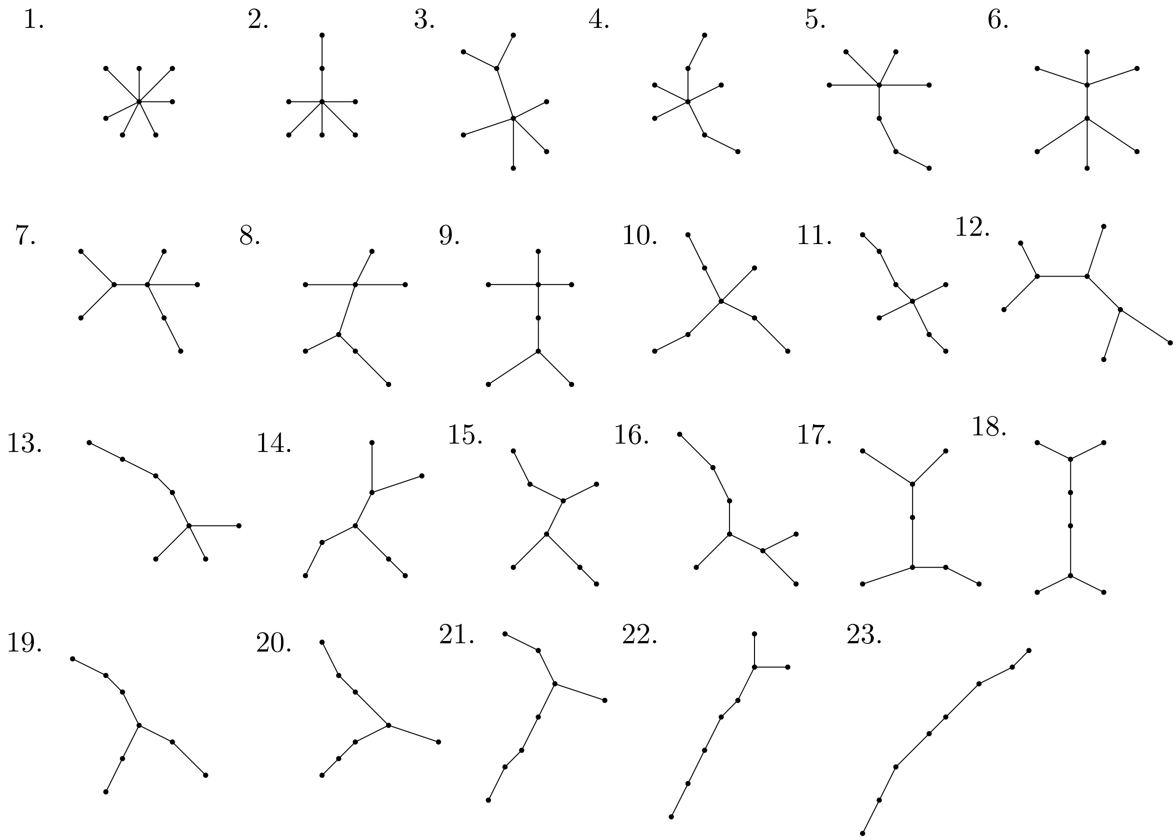

Fig. 2. The trees of order 8.

and thus

$$\xi_1 \geq \sqrt{\frac{T_{\mathcal{E}}^2(1) + T_{\mathcal{E}}^2(2) + \cdots + T_{\mathcal{E}}^2(n)}{\mathcal{E}^2(1) + \mathcal{E}^2(2) + \cdots + \mathcal{E}^2(n)}}.$$

Now assume that G is pseudo $\mathcal{E}$-regular. So $\frac{T_{\mathcal{E}}(i)}{\mathcal{E}(i)} = k$ for all i. It follows that $Y\mathcal{E} = kY$ and $k = \sqrt{\frac{T_{\mathcal{E}}^2(1)+T_{\mathcal{E}}^2(2)+\cdots+T_{\mathcal{E}}^2(n)}{\mathcal{E}^2(1)+\mathcal{E}^2(2)+\cdots+\mathcal{E}^2(n)}}$. Thus Y is an eigenvector corresponding to k, and so $\xi_1 = k$. Consequently the equality holds.

Conversely if equality holds then Y is the eigenvector corresponding to ξ_1, and that $Y\mathcal{E} = \xi_1 Y$. This then implies that $\frac{T_{\mathcal{E}}(i)}{\mathcal{E}(i)} = \xi_1$, or in other words G is pseudo $\mathcal{E}$-regular. □

5. Concluding remarks

Among the graph matrices the eccentricity matrix has an exotic behaviour. As we said the eccentricity matrix has recognized applications in Chemical Graph Theory. However, from a pure mathematical viewpoint, it would be interesting to find algebraic properties of the matrix which are related to the combinatorial structure of the graph. At this moment, it remains open the significant problem of characterizing the connected graphs whose eccentricity matrix is irreducible.

We now look back on Theorem 4.4 again. The equality holds in the theorem if and only if the graphs have two distinct $\mathcal{E}$-eigenvalues with the $\mathcal{E}$-spectral radius being simple. However, it is worth mentioning that a graph with two distinct $\mathcal{E}$-eigenvalues is not necessary to be a complete graph. Recall that the even cycle C_{2k} has two $\mathcal{E}$-eigenvalues $\pm k$ [26].

Naturally, an interesting research problem is put forward as follows.

Problem 1. Characterize the graphs with small number of distinct $\mathcal{E}$-eigenvalues.

For the adjacency matrix on this topic, we turn the readers to see van Dam's thesis [6] for more details. In the paper [27], we have investigated all the connected graphs with two distinct $\mathcal{E}$-eigenvalues.

Table 1
The $\mathcal{E}$-spectra of the trees of order 8.

1	12.5574	−0.5574	−2	−2	−2	−2	−2	−2	
	1	0	−87	−602	−1920	−3408	−3440	−1824	−384
2	8.2353	1.0861	0	0	0	0	−1.0861	−8.2353	
	1	0	−69	0	80	0	0	0	0
3	9.7287	1.1629	0	0	0	0	−1.1629	−9.7287	
	1	0	−96	0	128	0	0	0	0
4	10.6603	1.6056	0	0	0	0	−5.6056	−6.6603	
	1	0	−96	−248	639	0	0	0	0
5	11.6396	1.7541	0	0	0	0	−2.8402	−10.5535	
	1	0	−129	−128	612	0	0	0	0
6	10.1789	1.1789	0	0	0	0	−1.1789	−10.1789	
	1	0	−105	0	144	0	0	0	0
7	12.1456	1.7036	0	0	0	0	−5.5673	−8.2818	
	1	0	−125	−352	954	0	0	0	0
8	12.4485	1.7394	0	0	0	0	−4.5092	−9.6786	
	1	0	−136	−312	945	0	0	0	0
9	13.0507	1.8806	0	0	0	0	−3.1839	−11.7475	
	1	0	−161	−192	918	0	0	0	0
10	13.5394	1.6056	1.6056	0	0	−5.5394	−5.6056	−5.6056	
	1	0	−141	−656	807	4752	−6075	0	0
11	9.5609	3.9483	0	0	0	0	−3.9483	−9.5609	
	1	0	−107	0	1425	0	0	0	0
12	13.2736	1.831	0	0	0	0	−5.2736	−9.831	
	1	0	−152	−416	1260	0	0	0	0
13	12.7881	3.3861	0	0	0	0	−3.3861	−12.7881	
	1	0	−175	0	1875	0	0	0	0
14	14.6242	1.7421	1.6056	0	0	−4.1053	−5.6056	−8.2609	
	1	0	−168	−776	1191	5616	−7776	0	0
15	9.3739	4.3739	0	0	0	0	−4.3739	−9.3739	
	1	0	−107	0	1681	0	0	0	0
16	11.1597	4.0572	0	0	0	0	−4.0572	−11.1597	
	1	0	−141	0	2050	0	0	0	0
17	11.9426	3.7912	0	0	0	0	−3.7912	−11.9426	
	1	0	−157	0	2050	0	0	0	0
18	13.6603	3.6603	0	0	0	0	−3.6603	−13.6603	
	1	0	−200	0	2500	0	0	0	0
19	11.1597	4.0572	0	0	0	0	−4.0572	−11.1597	
	1	0	−141	0	2050	0	0	0	0
20	13	4.0711	0	0	0	0	−7	−10.0711	
	1	0	−168	−300	3731	0	0	0	0
21	12.0254	4.7769	0	0	0	0	−5.8112	−10.991	
	1	0	−161	−108	3669	0	0	0	0
22	14.6346	4.3806	0	0	0	0	−5.644	−13.3711	
	1	0	−222	−216	4838	0	0	0	0
23	12.9472	5.9472	0	0	0	0	−5.9472	−12.9472	
	1	0	−203	0	5929	0	0	0	0

We have shown that the eccentricity matrix of a tree is irreducible. Hence, the $\mathcal{E}$-spectral radii of trees are simple and their associated eigenvectors are non-negative. Along with other techniques, we expect to solve the next problem. See [26] for some initial discussions.

Problem 2. Which trees have the maximum $\mathcal{E}$-spectral radius?

About the least $\mathcal{E}$-eigenvalue $\xi_n(G)$ and the diameter of graphs, we put forward the following problem.

Problem 3. Determine the graphs with the least $\mathcal{E}$-eigenvalue $\xi_n(G) = -d$ $(d \geq 3)$.

6. Appendix: The $\mathcal{E}$-eigenvalues and $\mathcal{E}$-polynomials of trees

In Fig. 2 we depict the trees of order 8 and in Table 1 we list their $\mathcal{E}$-spectra. Those of trees of order at most 7 have been included in [26].

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