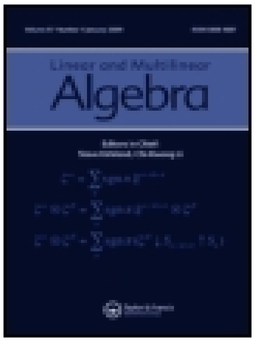

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Distance powers of integral Cayley graphs over dihedral groups and dicyclic groups

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ABSTRACT

In this paper, we focus on the dihedral groups and the dicyclic groups, and consider their corresponding integral Cayley graphs. We obtain the sufficient conditions for the integrality of the distance powers Γ^D of the Cayley graph $\Gamma = X(D_{2n}, S)$ (resp. $\Gamma = X(T_{4n}, S)$) ($n \geq 3$) for a set of nonnegative integers D. In particular, for a prime p, we show that if $\Gamma = X(D_{2p}, S)$ (resp. $\Gamma = X(T_{4p}, S)$) is integral, then the distance powers of $\Gamma = X(D_{2p}, S)$ (resp. $\Gamma = X(T_{4p}, S)$) are integral Cayley graphs.

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1. Introduction

Let $\Gamma = (V, E)$ be an undirected graph with vertex set V and edge set E. Its *adjacency matrix* is $A(\Gamma) = (a_{ij})_{n\times n}$, with $a_{ij} = 1$ if vertices i and j are adjacent in Γ, and $a_{ij} = 0$ otherwise. The *distance* between two vertices $x, y \in V$, denoted by $d(x, y)$, is the length of a shortest path connecting them.

Let D be a finite set of nonnegative integers. The *distance power* Γ^D of Γ is an undirected graph with vertex set V, two vertices x and y are adjacent in Γ^D if their distance $d(x, y)$ in Γ belongs to D. If Γ is not connected, it makes sense to allow $\infty \in D$. Clearly, $\Gamma^{\emptyset}$ is the graph without edges among V. The edge set of $\Gamma^{\{0\}}$ consists of a single loop at every vertex of Γ. If Γ has no loops, then $\Gamma^{\{1\}} = \Gamma$.

In this paper, we focus on a special class of regular graphs–Cayley graphs. Given a finite group G and a subset $1 \notin S \subseteq G$ with $S = S^{-1}$, the *Cayley graph* $X(G, S)$ has vertex set G and two vertices a, b are adjacent if $a^{-1}b \in S$. $X(G, S)$ is connected if S generates G.

We say that a graph Γ is *integral* if all eigenvalues of $A(\Gamma)$ are integers. In 1974, Harary and Schwenk [1] proposed the problem of classifying all integral graphs. In 2009, Ahmadi et al. [2] proved that the number of graphs on n vertices with integral spectrum is negligible compared to the total number of graphs. Thus, it seems plausible to classify all integral

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graphs. However, this turns out to be extremely hard and therefore this problem attracts much attention, one may see [3–6] for details. For an exhaustive survey in this subject, one may refer to [7]. In particular, Klotz and Sander [8] proved that for an abelian group G, $X(G, S)$ is integral if S belongs to the Boolean algebra generated by the subgroups of G. Alperin and Peterson [9] showed that the converse is true for all integral Cayley graphs over abelian groups. And Alperin [10] characterized the rational subsets of a finite group and discussed the relations with the integral Cayley graphs. In [11], Klotz and Sander further extended the previous results and proved that if Γ is an integral Cayley graph over the abelian group G, then every distance power Γ^D is also an integral Cayley graph over G for certain D.

The above results motivate us to generalize the results in [12] and consider the distance power of the integral Cayley graph over dihedral groups and dicyclic groups. This paper is organized as follows. At first, by using the Boolean algebra of the cyclic group $\langle a \rangle$, for a set D of nonnegative integers, we obtain some sufficient conditions for the integrality of a Cayley graph $\Gamma^D = X(D_{2n}, S^{(D)})$ $(n \geq 3)$ for a certain $S^{(D)}$ derived from S (see Theorem 3.4 and Corollaries 3.5 and 3.6). In particular, for an odd prime p, we show that if $\Gamma = X(D_{2p}, S)$ is integral, then every distance power of $\Gamma = X(D_{2p}, S)$ is integral (see Theorem 3.8). Finally, in Section 4, we obtain similar results about the distance power of integral Cayley graphs over the dicyclic groups T_{4n}.

2. Preliminaries

In this section, we will present some notation and several lemmas which will be used later.

We use $\mathbb{Z}_n$ to denote the additive cyclic group of order n, and $\mathbb{Z}_n^*$ is the multiplicative group of the units of the ring of integers modulo n. For a finite group G, we use $\mathrm{Aut}(G)$ to denote the group of the automorphisms of G.

A *representation* of a finite group G is a homomorphism $\rho : G \to GL(V)$ for some n-dimensional vector space over the complex field $\mathbb{C}$, where $GL(V)$ denotes the group of automorphisms of V. The dimension of V is called the *degree* of ρ. Two representations ρ_1 and ρ_2 of G on V_1 and V_2 respectively are *equivalent* if there is an isomorphism $T : V_1 \to V_2$ such that $T\rho_1(g) = \rho_2(g)T$ for all $g \in G$.

Let $\rho : G \to GL(V)$ be a representation. The *character* $\chi_\rho : G \to \mathbb{C}$ of ρ is defined by setting $\chi_\rho(g) = Tr(\rho(g))$ for $g \in G$, where $Tr(\rho(g))$ is the trace of the representation matrix of $\rho(g)$ for some basis of V. By the degree of χ_ρ we mean the degree of ρ which is simply $\chi_\rho(1)$. If W is a $\rho(g)$-invariant subspace of V for each $g \in G$, then we call W a *$\rho(G)$-invariant subspace* of V. If the only $\rho(G)$-invariant subspaces of V are $\{0\}$ and V, we call ρ an *irreducible representation* of G, and the corresponding character χ_ρ an *irreducible character* of G.

For a group G, we denote by $IRR(G)$ and $Irr(G)$ the complete set of non-equivalent irreducible representations of G and the complete set of non-equivalent irreducible characters of G, respectively.

The following lemma is crucial in our context.

Lemma 2.1 ([13]): *Let G be a finite group of order n, $S \subseteq G \setminus \{1\}$ be such that $S = S^{-1}$, and $Irr(G) = \{\chi_1, \ldots, \chi_h\}$ with $\chi_i(1) = d_i (i = 1, \ldots, h)$. Then the spectrum of the Cayley graph*

$X(G,S)$ can be arranged as

$$Spec(X(G,S)) = \left\{[\lambda_{11}]^{d_1}, \dots, [\lambda_{1d_1}]^{d_1}, \dots, [\lambda_{h1}]^{d_h}, \dots, [\lambda_{hd_h}]^{d_h}\right\}.$$

Furthermore, for any natural number t, we have

$$\lambda_{i1}^t + \lambda_{i2}^t + \cdots + \lambda_{id_i}^t = \sum_{s_1,\dots,s_t \in S} \chi_i\left(\prod_{l=1}^{t} s_l\right).$$

Let G be a finite group, and $\mathcal{F}_G$ be the set of all subgroups of G. Then the *Boolean algebra* $B(G)$ is the set whose elements are obtained as arbitrarily finite intersections, unions, and complements of the elements in $\mathcal{F}_G$. The minimal non-empty elements of $B(G)$ are called *atoms*. Each element of $B(G)$ is the union of some atoms, and atoms of $B(G)$ are the sets like $[g] = \{x \mid \langle x\rangle = \langle g\rangle, x \in G\}$.

For the cyclic group $\langle a\rangle$ of order n, the atoms of $B(\langle a\rangle)$ are the sets $[a^d] = \{a^l \mid gcd(l,n) = d\}$, where $d \mid n$. Then it is obtained

Lemma 2.2 ([14]): *Let $G = \langle a\rangle$ be the cyclic group of order n, and $[a^d]$ be one atom of $B(\langle a\rangle)$. Then $[a^d]^{-1} = [a^d]$. Furthermore, if $S \in B(\langle a\rangle)$, then $S = S^{-1}$.*

Lemma 2.3 ([11]): *If G is a finite abelian group with nonempty subsets $S, T \in B(G)$, then $ST \in B(G)$.*

By Lemmas 2.2 and 2.3, we have

Lemma 2.4: *Let $\langle a\rangle$ be the cyclic group with nonempty subsets $S, T \in B(\langle a\rangle)$. Then $ST = ST^{-1} \in B(\langle a\rangle)$.*

3. Distance powers of integral Cayley graphs over dihedral group

In this section, we focus on the dihedral group. The presentation for the dihedral group D_{2n} $(n \geq 3)$ is given by

$$D_{2n} = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1}\rangle.$$

This group has order $2n$, and

$$D_{2n} = \{a^k, a^k b \mid 0 \leq k \leq n-1\}.$$

Now we list the character table of D_{2n}.

Lemma 3.1 ([15]): *The character table of D_{2n} is given in Table 1 if n is odd, and in Table 2 otherwise, where ψ_i and χ_j are irreducible characters of degree one and two, respectively, and $1 \leq h \leq \lfloor\frac{n-1}{2}\rfloor$.*

Let A, B be two subsets of a group G. For any character χ of G, we denote $\chi(A) = \sum_{a\in A}\chi(a)$ and $\chi(AB) = \sum_{a\in A, b\in B}\chi(ab)$. Particularly, $\chi(A^2) = \sum_{a_1,a_2\in A}\chi(a_1a_2)$.

Table 1. Character table of D_{2n} for odd n.

	a^k	ba^k
ψ_1	1	1
ψ_2	1	-1
χ_h	$2\cos\left(\frac{2kh\pi}{n}\right)$	0

Table 2. Character table of D_{2n} for even n.

	a^k	ba^k
ψ_1	1	1
ψ_2	1	-1
ψ_3	$(-1)^k$	$(-1)^k$
ψ_4	$(-1)^k$	$(-1)^{k+1}$
χ_h	$2\cos\left(\frac{2kh\pi}{n}\right)$	0

Using the character table of D_{2n} and the Boolean algebra $B(\langle a\rangle)$ generated by the subgroups of cyclic group $\langle a\rangle$, Lu et al. [12] obtained several necessary and/or sufficient conditions for the integral Cayley graphs over dihedral group by Lemma 2.1.

Lemma 3.2 ([12]): *Let $D_{2n} = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1}\rangle$ be the dihedral group, and let $S = S_1 \cup S_2 \subseteq D_{2n}\backslash\{1\}$ be such that $S = S^{-1}$, where $S_1 \subseteq \langle a\rangle$ and $S_2 \subseteq b\langle a\rangle$. If $S_1, bS_2 \in B(\langle a\rangle)$, then $X(D_{2n}, S)$ is integral.*

Lemma 3.3 ([12]): *For an odd prime p, let $D_{2p} = \langle a, b \mid a^p = b^2 = 1, bab = a^{-1}\rangle$ and $S = S_1 \cup S_2$ be such that $1 \notin S, S = S^{-1}$, where $S_1 \subseteq \langle a\rangle, S_2 \subseteq b\langle a\rangle$. Then the Cayley graph $X(D_{2p}, S)$ is integral if and only if $S_1 = \emptyset$ or $\langle a\rangle \setminus \{1\}$ and $S_2 = b\langle a\rangle \setminus \{ba^j\}, b\langle a\rangle$ or $\{ba^j\}$ where $0 \leq j \leq p-1$.*

We concentrate on the connected Cayley graph over the dihedral group without loops, thus $0, \infty \notin D$ and $D \neq \emptyset$.

Theorem 3.4: *Let $S = S_1 \cup S_2 \subseteq D_{2n}\backslash\{1\}$ be such that $S = S^{-1}$ and $\langle S\rangle = D_{2n}$, where $S_1 \subseteq \langle a\rangle$ and $S_2 \subseteq b\langle a\rangle$. Let $\Gamma = X(D_{2n}, S)$ be the Cayley graph over D_{2n}. If $S_1, bS_2 \subseteq B(\langle a\rangle)$ and D is a set of positive integers, then the distance power $\Gamma^D = X(D_{2n}, S^{(D)})$ is an integral Cayley graph over D_{2n} for some $S^{(D)} \subseteq D_{2n}$.*

Proof: Firstly, we consider the case that D has only one element, say, $D = \{d\}$, we aim to find $S^{(d)} = S_1^{(d)} \cup S_2^{(d)}$ such that $\Gamma^{\{d\}} = X(D_{2n}, S^{(d)})$ is an integral Cayley graph over D_{2n}, where $S^{(d)} = (S^{(d)})^{-1}$.

If d is a number that is not the distance of any two vertices in Γ, then the assertion is confirmed as $\Gamma^{\{d\}} = X(D_{2n}, S^{(d)})$ with $S^{(d)} = \emptyset$.

If $d = 1$, then by Lemma 3.2, we have $\Gamma^{\{1\}} = \Gamma = X(D_{2n}, S)$ is an integral Cayley graph over D_{2n}.

Now assume that $d > 1$ is a finite distance of two vertices $x, y \in D_{2n}$. Using the relations $a^n = b^2 = 1, bab = a^{-1}$, then the last vertex y of a shortest path P connecting x and y in

Γ has the form

$$y = xa^{i_1} \cdots a^{i_k}(ba^{j_1}) \cdots (ba^{j_l}),$$

where $a^{i_1}, \ldots, a^{i_k} \in S_1$, $ba^{j_1}, \ldots, ba^{j_l} \in S_2$ and $k + l = d$. This implies

$$x^{-1}y = a^{i_1} \cdots a^{i_k}(ba^{j_1}) \cdots (ba^{j_l}) \in S^d,$$

where $S^d = \{s_1s_2 \ldots s_d \mid s_i \in S \text{ for } 0 \leq i \leq d\}$ is the d-fold product of the set S. To ensure that there is no path from x to y shorter than p, we need to remove from S^d all multiples S^m for $0 \leq m < d$ with $S^0 = \{1\}$. Now we consider

$$\begin{aligned}
S^{(d)} &= S^d \Big\backslash \bigcup_{0 \leqslant m < d} S^m \\
&= \left(\bigcup_{k+l=d} S_1^k \cdot S_2^l\right) \Big\backslash \left(\bigcup_{0 \leq m_1+m_2 < d} S_1^{m_1} \cdot S_2^{m_2}\right) \\
&= \bigcup_{k+l=d} \left(S_1^k \cdot S_2^l \Big\backslash \left(\bigcup_{0 \leq m_1+m_2 < d} S_1^{m_1} \cdot S_2^{m_2}\right)\right) \\
&= \bigcup_{k+l=d} \bigcap_{0 \leq m_1+m_2 < d} \left(S_1^k \cdot S_2^l \Big\backslash S_1^{m_1} \cdot S_2^{m_2}\right) \\
&= S_1^{(d)} \cup S_2^{(d)},
\end{aligned} \tag{1}$$

where

$$S_1^{(d)} = \bigcup_{\substack{k+l=d \\ l \text{is even}}} \bigcap_{0 \leq m_1+m_2 < d} \left(S_1^k \cdot S_2^l \Big\backslash S_1^{m_1} \cdot S_2^{m_2}\right)$$

and

$$S_2^{(d)} = \bigcup_{\substack{k+l=d \\ l \text{is odd}}} \bigcap_{0 \leq m_1+m_2 < d} \left(S_1^k \cdot S_2^l \Big\backslash S_1^{m_1} \cdot S_2^{m_2}\right).$$

Then we achieve $\Gamma^{\{d\}} = X(D_{2n}, S^{(d)})$, where $S^{(d)} = (S^{(d)})^{-1}$.

Note that, for each $ba^{j_1}ba^{j_2} \cdots ba^{j_t} \in S_2^t$, we have

$$ba^{j_1}ba^{j_2} \cdots ba^{j_t} = \begin{cases} a^{(j_2+j_4+\cdots+j_t)-(j_1+j_3+\cdots+j_{t-1})}, & t \text{ is even} \\ ba^{(j_1+j_3+\cdots+j_t)-(j_2+j_4+\cdots+j_{t-1})}, & t \text{ is odd} \end{cases}.$$

Since $bS_2 \in B(\langle a \rangle)$, we have $a^{j_i} \in B(\langle a \rangle)$ and thus, by Lemma 2.4, we obtain that $S_2^t \in B(\langle a \rangle)$ when t is even, and $bS_2^t \in B(\langle a \rangle)$ when t is odd.

Next we consider the following cases.

Case 1. l, m_2 are all even.

By the arguments above, we have $S_2^l, S_2^{m_2} \in B(\langle a \rangle)$, and thus $S_1^kS_2^l, S_1^{m_1}S_2^{m_2} \in B(\langle a \rangle)$. It leads to that $S_1^k \cdot S_2^l \backslash S_1^{m_1} \cdot S_2^{m_2} \in B(\langle a \rangle)$.

Case 2. l is even and m_2 is odd.

In this case, we have $S_2^l \in B(\langle a\rangle)$ and $bS_2^{m_2} \in B(\langle a\rangle)$. Therefore, $S_1^k S_2^l \in B(\langle a\rangle)$ and $S_1^{m_1} S_2^{m_2} \in bB(\langle a\rangle)$. It leads to

$$S_1^k \cdot S_2^l \Big\backslash S_1^{m_1} \cdot S_2^{m_2} = S_1^k \cdot S_2^l \in B(\langle a\rangle).$$

Combining Case 1 and Case 2, we obtain that $S_1^{(d)} \in B(\langle a\rangle)$. Similarly, one can obtain that $bS_2^{(d)} \in B(\langle a\rangle)$. Therefore, by Lemma 3.2, $\Gamma^{\{d\}} = X(D_{2n}, S^{(d)})$ is an integral Cayley graph over D_{2n}, where $S^{(d)} = S_1^{(d)} \cup S_2^{(d)}$.

Finally, we consider that D contains more elements. Let

$$D = \{d_1, d_2, \dots, d_v\}, \quad \text{and} \quad S_1^{(D)} = \bigcup_{i=1}^{v} S_1^{(d_i)}, \quad S_2^{(D)} = \bigcup_{i=1}^{v} S_2^{(d_i)},$$

where $d_1, d_2, \dots, d_v$ are nonnegative integers. Then we have $S_1^{(D)}, bS_2^{(D)} \in B(\langle a\rangle)$, and thus by Lemma 3.2 we get that the distance power $\Gamma^D = X(D_{2n}, S^{(D)})$ is an integral Cayley graph over D_{2n}, where $S^{(D)} = S_1^{(D)} \cup S_2^{(D)}$.

This completes the proof. ■

Let G be a finite group, and let $X(G, S)$ be a Cayley graph on G with respect to S. For any $\sigma \in \mathrm{Aut}(G)$, it is well known that σ induces an isomorphism Φ_σ from $X(G, S)$ to $X(G, \sigma(S))$, where Φ_σ is defined by $\Phi_\sigma(g) = \sigma(g)$ for $g \in G$. For the dihedral group D_{2n} $(n \geq 3)$, Rotmaler [16] proved that

$$\mathrm{Aut}\,(D_{2n}) = \left\{\sigma_{\lambda,k} \mid \lambda \in \mathbb{Z}_n^*, k \in \mathbb{Z}_n\right\},$$

where $\sigma_{\lambda,k}(a^i) = a^{\lambda i}$ and $\sigma_{\lambda,k}(ba^j) = ba^{\lambda j+k}$ for $a^i, ba^j \in D_{2n}$. From the above arguments and Theorem 3.4, we obtain

Corollary 3.5: *Let $S = S_1 \cup S_2 \subseteq D_{2n}\backslash\{1\}$ be such that $S = S^{-1}$ and $\langle S\rangle = D_{2n}$, where $S_1 \subseteq \langle a\rangle$ and $S_2 \subseteq b\langle a\rangle$, and let $\Gamma = X(D_{2n}, S)$ be a Cayley graph over D_{2n}. If $S_1 \subseteq B(\langle a\rangle)$, $|S_2| = n - 1$ and D is a set of positive integers, then $\Gamma^D = X(D_{2n}, S^{(D)})$ is an integral Cayley graph over D_{2n}.*

Proof: Since $|S_2| = n - 1$, we may assume that $S_2 = b\langle a\rangle \setminus \{ba^j\}$ for some $0 \leq j \leq n - 1$.

If $S_2 = b\langle a\rangle \setminus \{b\}$, then $bS_2 = \langle a\rangle \setminus \{1\} \in B(\langle a\rangle)$, and $S_1 \subseteq B(\langle a\rangle)$. By Theorem 3.4, the distance power Γ^D is an integral Cayley graph over D_{2n}.

If $S_2 = b\langle a\rangle \setminus \{ba^j\}$, for some $1 \leq j \leq n - 1$ and $n \geq 3$, then $\sigma_{1,j} \in \mathrm{Aut}(D_{2n})$ such that $\sigma_{1,j}(a^i) = a^i$ and $\sigma_{1,j}(b) = ba^j$. Moreover, $\sigma_{1,j}(S_1) = S_1$ and $\sigma_{1,j}(b\langle a\rangle \setminus \{b\}) = b\langle a\rangle \setminus \{ba^j\}$. Therefore, we have $\Gamma = X(D_{2n}, S_1 \cup (b\langle a\rangle \setminus \{ba^j\})) \cong X(D_{2n}, S_1 \cup (b\langle a\rangle \setminus \{b\}))$. Thus Γ^D is an integral Cayley graph over D_{2n}.

This completes the proof. ■

Noticing that $\Gamma = X(D_{2n}, S_1 \cup \{b\}) \cong \Gamma' = X(D_{2n}, S_1 \cup \{ba^i\})$ for any $1 \leq i \leq n - 1$, we obtain the following result similar to Corollary 3.5.

Corollary 3.6: *Let $S = S_1 \cup S_2 \subseteq D_{2n}\backslash\{1\}$ be such that $S = S^{-1}$ and $\langle S\rangle = D_{2n}$, where $S_1 \subseteq \langle a\rangle$ and $S_2 \subseteq b\langle a\rangle$, and let $\Gamma = X(D_{2n}, S)$ be a Cayley graph over D_{2n}. If $S_1 \subseteq$*

$B(\langle a\rangle), |S_2| = 1$ and D is a set of positive integers, then $\Gamma^D = X(D_{2n}, S^{(D)})$ is an integral Cayley graph over D_{2n}.

From Theorem 3.4 (for $S_1, bS_2 \subseteq B(\langle a\rangle)$), Corollary 3.6 (for $S_1 \subseteq B(\langle a\rangle), |S_2| = 1$) and Corollary 3.5 ($S_1 \subseteq B(\langle a\rangle)$, $|S_2| = n-1$), we get the following corollary for the dihedral group D_{2p}, where p is an odd prime.

Corollary 3.7: *For an odd prime p, let $D_{2p} = \langle a, b \mid a^p = b^2 = 1, bab = a^{-1}\rangle$ and $S = S_1 \cup S_2$ be such that $1 \notin S, S = S^{-1}$, where $S_1 \subseteq \langle a\rangle$, $S_2 \subseteq b\langle a\rangle$. Let D be a set of positive integers. Let $\Gamma = X(D_{2p}, S)$ be a Cayley graph over D_{2p}. If $S_1 = \emptyset$ or $\langle a\rangle \setminus \{1\}$ and $S_2 = b\langle a\rangle \setminus \{ba^j\}$, $b\langle a\rangle$ or $\{ba^j\}$ where $0 \le j \le p-1$, then $\Gamma^D = X(D_{2p}, S^{(D)})$ is an integral Cayley graph over D_{2p}.*

Thus, from Lemma 3.3 and Corollary 3.7, we obtain a sufficient condition for the integrality of the Cayley graph $\Gamma^D = X(D_{2p}, S^{(D)})$.

Theorem 3.8: *For an odd prime p, let $D_{2p} = \langle a, b \mid a^p = b^2 = 1, bab = a^{-1}\rangle$ and $S = S_1 \cup S_2$ be such that $1 \notin S$, $S = S^{-1}$, where $S_1 \subseteq \langle a\rangle$, $S_2 \subseteq b\langle a\rangle$. If the Cayley graph $X(D_{2p}, S)$ is integral, and if D is a set of positive integers, then $\Gamma^D = X(D_{2p}, S^{(D)})$ is an integral Cayley graph over D_{2p} for certain $S^{(D)}$.*

We would like to point out that the necessity of Theorem 3.8 is not true. For example, we take the set $D = \{d\}$, where d is a number that is not the distance of any two vertices in Γ, and let $S = \{a, a^{-1}\} \cup \{b\}$. Then $\Gamma^D = X(D_{2p}, S^{(D)})$ with $S^{(D)} = \emptyset$ is integral, but $\Gamma = X(D_{2p}, S)$ is not integral from Lemma 3.3.

4. Distance powers of integral Cayley graphs over dicyclic group

In this section, we consider the dicyclic group, which is the metacyclic group of order $4n$ for odd n [17]. The presentation for the dicyclic group T_{4n} ($n \ge 3$) is given by

$$T_{4n} = \langle a, b \mid a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1}\rangle.$$

This group has order $4n$, and

$$T_{4n} = \{a^k, ba^k \mid 0 \le k \le 2n-1\}.$$

In [18], we gave several criteria for the integral Cayley graph over the dicyclic group T_{4n}.

Lemma 4.1 ([18]): *Let $T_{4n} = \langle a, b \mid a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1}\rangle$ be the dicyclic group, and let $S = H_1 \cup H_2 \subseteq T_{4n} \setminus \{1\}$ be such that $S = S^{-1}$, where $H_1 \subseteq \langle a\rangle$ and $H_2 \subseteq b\langle a\rangle$. If $H_1, bH_2 \in B(\langle a\rangle)$, then $X(T_{4n}, S)$ is integral.*

Lemma 4.2 ([18]): *For an odd prime p, let $T_{4p} = \langle a, b | a^{2p} = 1, a^p = b^2, b^{-1}ab = a^{-1}\rangle$ and $S = H_1 \cup H_2$ be such that $S = S^{-1}$, where $H_1 \subseteq \langle a\rangle$ and $H_2 \subseteq b\langle a\rangle$. Then $X(T_{4p}, S)$ is integral if and only if $H_1 \in B(\langle a\rangle)$ which has four atoms $\{\{1\}, [a], [a^2], [a^p]\}$ and $H_2 = b\langle a\rangle \setminus \{ba^k, ba^{p+k}\}, b\langle a\rangle$ or $\{ba^k, ba^{p+k}\}$, where $0 \le k \le p-1$.*

The automorphism group of T_{4n} is very important for our study.

Lemma 4.3: *Let $T_{4n} = \langle a, b \mid a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1} \rangle$ be the dicyclic group. Then* $\mathrm{Aut}(T_{4n}) = \{\alpha_{g,h} \mid g \in \mathbb{Z}_{2n}^*, h \in \mathbb{Z}_{2n}\}$, *where* $\alpha_{g,h}(a^k) = a^{kg}, \alpha_{g,h}(ba^k) = ba^{kg+h}$.

Proof: Note that $\alpha \in \mathrm{Aut}(T_{4n})$ if and only if $o(\alpha(a)) = o(a) = 2n$, $\alpha(a)^n = \alpha(b)^2$, and $\alpha(b)^{-1}\alpha(a)\alpha(b) = \alpha(a)^{-1}$. Therefore,

$$\alpha(a) \in \{a^k \mid 0 \le k \le 2n-1, \text{ and } (k, 2n) = 1\},$$
$$\alpha(b) \in \{ba^k \mid 0 \le k \le 2n-1\}.$$

Hence, $|\mathrm{Aut}(T_{4n})| \le \varphi(2n) \times 2n$, where φ is the Euler's function. We now define the function

$$\alpha_{g,h} : T_{4n} \mapsto T_{4n}$$

such that

$$\alpha_{g,h}(a^k) = a^{kg}, \quad \alpha_{g,h}(ba^k) = ba^{kg+h},$$

where $g \in \mathbb{Z}_{2n}^*$ and $h \in \mathbb{Z}_{2n}$. Then $\alpha_{g,h}$ is a bijection, and the homomorphism is as follows.

$$\begin{aligned}
\alpha_{g,h}(a^{k_1}a^{k_2}) &= \alpha_{g,h}(a^{k_1+k_2}) = a^{(k_1+k_2)g} = a^{k_1g+k_2g} = a^{k_1g}a^{k_2g} = \alpha_{g,h}(a^{k_1})\alpha_{g,h}(a^{k_2}),\\
\alpha_{g,h}(ba^{k_1}ba^{k_2}) &= \alpha_{g,h}(a^{-k_1+k_2+n}) = a^{(-k_1+k_2+n)g}\\
&= a^{-k_1g+k_2g+n} \quad (\text{because } (g,2n)=1, \text{ then } ng \equiv n \pmod{2n})\\
&= a^{-(k_1g+h)+(k_2g+h)+n} = ba^{k_1g+h}ba^{k_2g+h} = \alpha_{g,h}(ba^{k_1})\alpha_{g,h}(ba^{k_2}),\\
\alpha_{g,h}(a^{k_1}ba^{k_2}) &= \alpha_{g,h}(ba^{k_2-k_1}) = ba^{(k_2-k_1)g+h} = a^{k_1g}ba^{k_2g+h} = \alpha_{g,h}(a^{k_1})\alpha_{g,h}(ba^{k_2}),\\
\alpha_{g,h}(ba^{k_1}a^{k_2}) &= \alpha_{g,h}(ba^{k_1+k_2}) = ba^{(k_1+k_2)g+h} = ba^{k_1g+h}a^{k_2g} = \alpha_{g,h}(ba^{k_1})\alpha_{g,h}(a^{k_2}).
\end{aligned}$$

Moreover, using these equations above, we have

$$\alpha_{g,h}(a)^{2n} = \alpha_{g,h}(a^{2n}) = \alpha_{g,h}(1) = 1,$$
$$\alpha_{g,h}(a)^n = \alpha_{g,h}(a^n) = \alpha_{g,h}(b^2) = \alpha_{g,h}(b)^2,$$

and

$$\alpha_{g,h}(a)\alpha_{g,h}(b)\alpha_{g,h}(a) = \alpha_{g,h}(aba) = \alpha_{g,h}(b),$$

i.e.

$$\alpha_{g,h}(b)^{-1}\alpha_{g,h}(a)\alpha_{g,h}(b) = \alpha_{g,h}(a)^{-1}.$$

Therefore, $\alpha_{g,h} \in \mathrm{Aut}(T_{4n})$. Since $|\mathrm{Aut}(T_{4n})| \le \varphi(2n) \times 2n$ and $|\{\alpha_{g,h} \mid g \in \mathbb{Z}_{2n}^*, h \in \mathbb{Z}_{2n}\}| = \varphi(2n) \times 2n$, we have $\mathrm{Aut}(T_{4n}) = \{\alpha_{g,h} \mid g \in \mathbb{Z}_{2n}^*, h \in \mathbb{Z}_{2n}\}$.

This completes the proof. ■

Now, along the way of Section 3, we can obtain similar conclusions about the distance powers of integral Cayley graphs over the dicyclic group T_{4n}.

Theorem 4.4: *Let $S = H_1 \cup H_2 \subseteq T_{4n}\backslash\{1\}$ be such that $S = S^{-1}$ and $\langle S\rangle = T_{4n}$, where $H_1 \subseteq \langle a\rangle$ and $H_2 \subseteq b\langle a\rangle$. Let $\Gamma = X(T_{4n}, S)$ be the Cayley graph over T_{4n}. If $H_1, bH_2 \subseteq B(\langle a\rangle)$ and D is a set of positive integers, then the distance power $\Gamma^D = X(T_{4n}, S^{(D)})$ is an integral Cayley graph over T_{4n} for some $S^{(D)} \subseteq T_{4n}$.*

Proof: By Lemma 4.1, similar to the proof of Theorem 3.4. ■

Corollary 4.5: *Let $S = H_1 \cup H_2 \subseteq T_{4n}\backslash\{1\}$ be such that $S = S^{-1}$ and $\langle S\rangle = T_{4n}$, where $H_1 \subseteq \langle a\rangle$ and $H_2 \subseteq b\langle a\rangle$, and let $\Gamma = X(T_{4n}, S)$ be a Cayley graph over T_{4n}. If $H_1 \subseteq B(\langle a\rangle)$, $|H_2| = 2$ and D is a set of positive integers, then $\Gamma^D = X(T_{4n}, S^{(D)})$ is an integral Cayley graph over T_{4n}.*

Proof: Since $H_2^{-1} = H_2$ and $|H_2| = 2$, we may assume that $H_2 = \{ba^j, (ba^j)^{-1} = ba^{n+j}\}$ for some $0 \le j \le n-1$.

If $H_2 = \{b, ba^n\}$, then $bH_2 = \{1, a^n\} \in B(\langle a\rangle)$, and $H_1 \subseteq B(\langle a\rangle)$. By Theorem 4.4, the distance power Γ^D is an integral Cayley graph over T_{4n}.

If $H_2 = \{ba^j, (ba^j)^{-1} = ba^{n+j}\}$ for some $1 \le j \le n-1$. Since $1 \in \mathbb{Z}_{2n}^*$ and $j \in \mathbb{Z}_{2n}$, we have $\alpha_{1,j} \in \mathrm{Aut}(T_{4n})$. Then $\alpha_{1,j}(a^k) = a^k$, $\alpha_{1,j}(b) = ba^j$ and $\alpha_{1,j}(b) = ba^{n+j}$. Moreover, $\alpha_{1,j}(H_1) = H_1$. Therefore, we have $\Gamma = X(T_{4n}, H_1 \cup \{ba^j, ba^{n+j}\}) \cong X(T_{4n}, H_1 \cup \{b, ba^n\})$. Thus Γ^D is an integral Cayley graph over T_{4n}.

This completes the proof. ■

Similar as Corollary 4.5, we obtain

Corollary 4.6: *Let $S = H_1 \cup H_2 \subseteq D_n\backslash\{1\}$ be such that $S = S^{-1}$ and $\langle S\rangle = T_{4n}$, where $H_1 \subseteq \langle a\rangle$ and $H_2 \subseteq b\langle a\rangle$, and let $\Gamma = X(T_{4n}, S)$ be a Cayley graph over T_{4n}. If $H_1 \subseteq B(\langle a\rangle)$, $|H_2| = 2n-2$ and D is a set of positive integers, then $\Gamma^D = X(T_{4n}, S^{(D)})$ is an integral Cayley graph over T_{4n}.*

Thus, from Lemma 4.2 and Corollaries 4.5 and 4.6, we obtain the following theorem for dicyclic group T_{4p}, where p is an odd prime.

Theorem 4.7: *For an odd prime p, let $T_{4p} = \langle a, b \mid a^{2p} = 1, a^p = b^2, b^{-1}ab = a^{-1}\rangle$ and $S = H_1 \cup H_2$ such that $1 \notin S, S = S^{-1}$, where $H_1 \subseteq \langle a\rangle$, $H_2 \subseteq b\langle a\rangle$. If the Cayley graph $X(T_{4p}, S)$ is integral, and if D be a set of positive integers, then $\Gamma^D = X(T_{4p}, S^{(D)})$ is an integral Cayley graph over T_{4p} for certain $S^{(D)}$.*

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