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Short Communication

A short remark on graphs with two main eigenvalues

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ABSTRACT

We give a short proof of a conjecture on graphs with two main eigenvalues posed by Réti in [Applied Mathematics and Computation 344–345 (2019), 107–115].

Let $G = (V, E)$ be a simple graph. The spectrum of G is defined as the spectrum of its (0,1)-adjacency matrix A. A *main eigenvalue* of G is any eigenvalue λ of A whose eigenspace is not orthogonal to the all-one vector $\mathbf{j}$. The largest eigenvalue of A is always the main eigenvalue of G, as it has an eigenvector that is positive on at least one component of G by the Perron–Frobenius theorem [1, Theorem 1.3.6]. Peter Rowlinson [4, Proposition 1.4] showed that only regular graphs have exactly one main eigenvalue, and proved the following result.

Theorem 1 [4]**.** *Graph G and its complement $\overline{G}$ have the same number of main eigenvalues.*

Hagos [2] characterized the number of the main eigenvalues of a graph in terms of the walk vectors.

Theorem 2 [2]**.** *If k is the maximal integer such that $\mathbf{j}$, $A\mathbf{j}$, ..., $A^{k-1}\mathbf{j}$ are linearly independent, then G has exactly k main eigenvalues.*

Réti [3] introduced *complete split-like* graphs recently, by defining a complete split-like graph $KSL(n, q, \delta)$ as an n-vertex graph with $q \geq 1$ vertices of degree $n-1$ and $n-q$ vertices of degree δ, where $q \leq \delta < n-1$, and then posed the following conjecture.

Conjecture 1 [3]**.** The complete split-like graph *$KSL(n, q, \delta)$ has exactly two main eigenvalues.*

We show here that this conjecture follows easily from both Theorems 1 and 2.

Let U denote the set of vertices of degree $n-1$, and V the set of vertices of degree δ in $KSL(n, q, \delta)$. Note that each vertex of V is adjacent to all q vertices of U, so that it is adjacent to $\delta - q$ other vertices of V. Hence the complement $\overline{KSL(n, q, \delta)}$ is the union of a $(n-\delta-1)$-regular graph H on the set of vertices V and the isolated vertices from U. The spectrum of

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$\overline{KSL(n,q,\delta)}$ is the union of the spectrum of H, with the corresponding eigenvectors of H extended with zero components for vertices in U, and q zeros, with a corresponding eigenvector $\mathbf{e}_u$ for each vertex $u \in U$, that has component 1 for u and 0 for the remaining vertices. It is then clear that $\overline{KSL(n,q,\delta)}$ has exactly two main eigenvalues: the spectral radius $n-\delta-1$ of the regular graph H and the eigenvalue 0, as $\mathbf{e}_u^\top \mathbf{j} \neq 0$. Conjecture 1 then follows from Theorem 1.

Elias Hagos' characterization from Theorem 2 shows that a graph has exactly two main eigenvalues if it is not regular and there exists real values α and β such that

$$A^2\mathbf{j} = \alpha A\mathbf{j} + \beta\mathbf{j}.$$

Let A denote the adjacency matrix of $KSL(n, q, \delta)$ and let d_z denote the degree of vertex $z \in U \cup V$. We have

$$A\mathbf{j} = (d_z)_{z\in U\cup V} \quad \text{and} \quad A^2\mathbf{j} = \left(\sum_{w\sim z} d_w\right)_{z\in U\cup V},$$

where $w \sim z$ denotes that w and z are adjacent. Let $\mathbf{j}_U$ denote the all-one vector with the set of components U and let $\mathbf{j}_V$ denote the all-one vector with the set of components V. Then

$$\mathbf{j} = \begin{bmatrix} \mathbf{j}_U \\ \mathbf{j}_V \end{bmatrix}, \qquad A\mathbf{j} = \begin{bmatrix} (n-1)\mathbf{j}_U \\ \delta\mathbf{j}_V \end{bmatrix} \quad \text{and} \quad A^2\mathbf{j} = \begin{bmatrix} ((q-1)(n-1)+(n-q)\delta)\mathbf{j}_U \\ (q(n-1)+(\delta-q)\delta)\mathbf{j}_V \end{bmatrix},$$

so that

$$A^2\mathbf{j} = (\delta-1)A\mathbf{j} + (q(n-\delta-1)+\delta)\mathbf{j}.$$

Hence j, Aj and A^2j are linearly dependent, while j and Aj are linearly independent as $\delta < n-1$, so that Conjecture 1 now follows by Theorem 2. By [2, Corollary 2.5] the main eigenvalues of $KSL(n, q, \delta)$ are equal to

$$\lambda, \mu = \frac{\alpha \pm \sqrt{\alpha^2+4\beta}}{2} = \frac{\delta - 1 \pm \sqrt{(\delta+1)^2 + 4q(n-\delta-1)}}{2}.$$

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