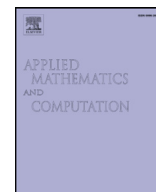




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## Short Communication

## A short remark on graphs with two main eigenvalues

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## ABSTRACT

We give a short proof of a conjecture on graphs with two main eigenvalues posed by Réti in [Applied Mathematics and Computation 344–345 (2019), 107–115].

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Let  $G = (V, E)$  be a simple graph. The spectrum of  $G$  is defined as the spectrum of its  $(0,1)$ -adjacency matrix  $A$ . A *main eigenvalue* of  $G$  is any eigenvalue  $\lambda$  of  $A$  whose eigenspace is not orthogonal to the all-one vector  $\mathbf{j}$ . The largest eigenvalue of  $A$  is always the main eigenvalue of  $G$ , as it has an eigenvector that is positive on at least one component of  $G$  by the Perron–Frobenius theorem [1, Theorem 1.3.6]. Peter Rowlinson [4, Proposition 1.4] showed that only regular graphs have exactly one main eigenvalue, and proved the following result.

**Theorem 1** [4]. *Graph  $G$  and its complement  $\bar{G}$  have the same number of main eigenvalues.*

Hagos [2] characterized the number of the main eigenvalues of a graph in terms of the walk vectors.

**Theorem 2** [2]. *If  $k$  is the maximal integer such that  $\mathbf{j}, A\mathbf{j}, \dots, A^{k-1}\mathbf{j}$  are linearly independent, then  $G$  has exactly  $k$  main eigenvalues.*

Réti [3] introduced *complete split-like* graphs recently, by defining a complete split-like graph  $KSL(n, q, \delta)$  as an  $n$ -vertex graph with  $q \geq 1$  vertices of degree  $n - 1$  and  $n - q$  vertices of degree  $\delta$ , where  $q \leq \delta < n - 1$ , and then posed the following conjecture.

**Conjecture 1** [3]. *The complete split-like graph  $KSL(n, q, \delta)$  has exactly two main eigenvalues.*

We show here that this conjecture follows easily from both Theorems 1 and 2.

Let  $U$  denote the set of vertices of degree  $n - 1$ , and  $V$  the set of vertices of degree  $\delta$  in  $KSL(n, q, \delta)$ . Note that each vertex of  $V$  is adjacent to all  $q$  vertices of  $U$ , so that it is adjacent to  $\delta - q$  other vertices of  $V$ . Hence the complement  $\bar{KSL}(n, q, \delta)$  is the union of a  $(n - \delta - 1)$ -regular graph  $H$  on the set of vertices  $V$  and the isolated vertices from  $U$ . The spectrum of

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$\overline{KSL}(n, q, \delta)$  is the union of the spectrum of  $H$ , with the corresponding eigenvectors of  $H$  extended with zero components for vertices in  $U$ , and  $q$  zeros, with a corresponding eigenvector  $\mathbf{e}_u$  for each vertex  $u \in U$ , that has component 1 for  $u$  and 0 for the remaining vertices. It is then clear that  $\overline{KSL}(n, q, \delta)$  has exactly two main eigenvalues: the spectral radius  $n - \delta - 1$  of the regular graph  $H$  and the eigenvalue 0, as  $\mathbf{e}_u^\top \mathbf{j} \neq 0$ . [Conjecture 1](#) then follows from [Theorem 1](#).

Elias Hagos' characterization from [Theorem 2](#) shows that a graph has exactly two main eigenvalues if it is not regular and there exists real values  $\alpha$  and  $\beta$  such that

$$A^2 \mathbf{j} = \alpha A \mathbf{j} + \beta \mathbf{j}.$$

Let  $A$  denote the adjacency matrix of  $\overline{KSL}(n, q, \delta)$  and let  $d_z$  denote the degree of vertex  $z \in U \cup V$ . We have

$$A \mathbf{j} = (d_z)_{z \in U \cup V} \quad \text{and} \quad A^2 \mathbf{j} = \left( \sum_{w \sim z} d_w \right)_{z \in U \cup V},$$

where  $w \sim z$  denotes that  $w$  and  $z$  are adjacent. Let  $\mathbf{j}_U$  denote the all-one vector with the set of components  $U$  and let  $\mathbf{j}_V$  denote the all-one vector with the set of components  $V$ . Then

$$\mathbf{j} = \begin{bmatrix} \mathbf{j}_U \\ \mathbf{j}_V \end{bmatrix}, \quad A \mathbf{j} = \begin{bmatrix} (n-1)\mathbf{j}_U \\ \delta \mathbf{j}_V \end{bmatrix} \quad \text{and} \quad A^2 \mathbf{j} = \begin{bmatrix} ((q-1)(n-1) + (n-q)\delta)\mathbf{j}_U \\ (q(n-1) + (\delta-q)\delta)\mathbf{j}_V \end{bmatrix},$$

so that

$$A^2 \mathbf{j} = (\delta - 1)A \mathbf{j} + (q(n - \delta - 1) + \delta)\mathbf{j}.$$

Hence  $\mathbf{j}$ ,  $A \mathbf{j}$  and  $A^2 \mathbf{j}$  are linearly dependent, while  $\mathbf{j}$  and  $A \mathbf{j}$  are linearly independent as  $\delta < n - 1$ , so that [Conjecture 1](#) now follows by [Theorem 2](#). By [2, Corollary 2.5] the main eigenvalues of  $\overline{KSL}(n, q, \delta)$  are equal to

$$\lambda, \mu = \frac{\alpha \pm \sqrt{\alpha^2 + 4\beta}}{2} = \frac{\delta - 1 \pm \sqrt{(\delta + 1)^2 + 4q(n - \delta - 1)}}{2}.$$

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