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Note

A short note on the sum of k largest distance eigenvalues of bipartite graphs

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ABSTRACT

For a connected graph of order n and an integer $k \geq 1$, denoted by $S_k(D(G)) = \lambda_1(D(G)) + \cdots + \lambda_k(D(G))$ the sum of the k largest distance eigenvalues of G. In this note, by using a basic technique, we show that $\lambda_1(D(G)) + \lambda_2(D(G)) \geq 2n - 4$ when G is bipartite. It leads to that $S_k(D(G)) \geq 2n - 2k$ when n is large enough with respect to k, which completely settles a problem posed by Lin (2019).

1. Introduction

Let G be a connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$, where $|V(G)| = n$ and $|E(G)| = m$. The *distance* between vertices v_i and v_j, denoted by $d_G(v_i, v_j)$ (or $d(v_i, v_j)$ for short), is the length of a shortest path from v_i to v_j in G. The *distance matrix* of G, denoted by $D(G)$(or simple by D), is the real symmetric matrix with (i, j)-entry being $d_G(v_i, v_j)$. The *distance eigenvalues* of G, are denoted by $\lambda_1(D(G)) \geq \lambda_2(D(G)) \geq \cdots \geq \lambda_n(D(G))$. For more details about the distance matrix we refer the reader to [1].

For a number $2 \leq k \leq n - 1$, denote by $S_k(D(G)) = \lambda_1(D(G)) + \lambda_2(D(G)) + \cdots + \lambda_k(D(G))$. In 2019, Lin [3] determined the sharp lower bounds of $S_k(D(G))$ when G is a connected graph and a tree, respectively, and left the following problem.

Problem 1 (*[3, Problem 3]*)**.** Let G be a connected bipartite graph of order n. For an integer $k \geq 2$ and sufficiently large n with respect to k, does there always hold $S_k(D(G)) \geq 2n - 2k$, where the equality holds if and only if $G = K_{r,n-r}$ for $1 \leq r \leq n - 1$?

Subsequently, Lin and Zhang [4] show that $S_k(D(G)) \geq 2n - 2k$ if G is a C_4-free bipartite graph or a bipartite distance regular graph. This result partially solved the above problem. In this short note, we settle this problem by proving $\lambda_1(D(G)) + \lambda_2(D(G)) \geq 2n - 4$ when G is a connected bipartite graph on n vertices.

2. Lemmas and results

Firstly, we will list some preliminaries and prove some lemmas. The following lemma is our main tool, which helps us obtain the lower bound of the sum of eigenvalues.

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Lemma 1 (*[2, Theorem 1]*). *Let M be a symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then*

$$\lambda_1 + \lambda_2 + \cdots + \lambda_r = \sup\{u_1^T M u_1 + u_2^T M u_2 + \cdots + u_r^T M u_r\} (r = 1, 2, \ldots, n),$$

where the supremum is taken over all orthonormal vectors $u_1, u_2, \ldots, u_r$.

Now we are in the position to present our main result.

Theorem 1. *Let G be a connected bipartite graph of order n. Then $S_2(D(G)) \geq 2n - 4$, where the equality holds if and only if $G = K_{r,n-r}$ for $1 \leq r \leq n-1$.*

Proof. By simple calculations, we have $\lambda_1(D(K_{r,n-r})) = n - 2 + \sqrt{n^2 + 3r^2 - 3nr}$ and $\lambda_2(D(K_{r,n-r})) = n - 2 - \sqrt{n^2 + 3r^2 - 3nr}$ for $1 \leq r \leq n-1$. Therefore, $S_2(D(K_{r,n-r})) = 2n - 4$.

Let $G = G[V_1, V_2]$ be a connected bipartite graph with $|V_1| = r$ and $|V_2| = n - r$ but $G \neq K_{r,n-r}$. Let

$$u_1 = \frac{1}{\sqrt{r}}(\underbrace{1, \ldots, 1}_{r}, \underbrace{0, \ldots, 0}_{n-r})^T$$

and

$$u_2 = \frac{1}{\sqrt{n-r}}(\underbrace{0, \ldots, 0}_{r}, \underbrace{-1, \ldots, -1}_{n-r})^T.$$

It is easy to see that $u_1^T u_1 = u_2^T u_2 = 1$ and $u_1^T u_2 = 0$, that is, u_1 and u_2 are orthonormal vectors. From Lemma 1, we have

$$\begin{aligned}
& \lambda_1(D(G)) + \lambda_2(D(G)) \geq u_1^T D(G) u_1 + u_2^T D(G) u_2 \\
= \; & \textstyle\sum_{v_i, v_j} d(v_i, v_j) u_1(i) u_1(j) + \sum_{v_i, v_j} d(v_i, v_j) u_2(i) u_2(j) \\
= \; & \textstyle\sum_{v_i, v_j \in V_1} d(v_i, v_j)\,(u_1(i)u_1(j) + u_2(i)u_2(j)) + \sum_{v_i, v_j \in V_2} d(v_i, v_j)\,(u_1(i)u_1(j) + u_2(i)u_2(j)) \\
& \textstyle + 2\sum_{v_i \in V_1, v_j \in V_2} d(v_i, v_j)\,(u_1(i)u_1(j) + u_2(i)u_2(j)) \\
= \; & \textstyle\sum_{v_i, v_j \in V_1} d(v_i, v_j) \frac{1}{r} + \sum_{v_i, v_j \in V_2} d(v_i, v_j) \frac{1}{n-r} \\
> \; & 2r(r-1)\tfrac{1}{r} + 2(n-r)(n-r-1)\tfrac{1}{n-r} \\
= \; & 2n - 4,
\end{aligned}$$

where the second to last inequality holds because G is not complete bipartite and thus there exist two vertices v_s, v_t in a same part such that $d(v_s, v_t) > 2$ and all other pairs (x, y) in a same part satisfy $d(x, y) \geq 2$.

The proof is completed. □

Lemma 2 (*[3, Lemma 3]*). *Let G be a connected graph of order n. For any integer $k \geq 2$, if n is sufficiently large with respect to k then $\lambda_k(D(G)) \geq -2$.*

Theorem 2. *Let G be a connected bipartite graph of order n. For an integer $k \geq 2$ and sufficiently large n with respect to k, we have $S_k(D(G)) \geq 2n - 2k$, where the equality holds if and only if $G = k_{r,n-r}$ for $1 \leq r \leq n-1$.*

Proof. It is known that $S_k(D(K_{r,n-r})) = 2n - 2k$ for $k \geq 2$. Let $G = G[V_1, V_2]$ be a connected bipartite with $|V_1| = r$ and $|V_2| = n - r$ but $G \neq K_{r,n-r}$. Then, by Theorem 1 and Lemma 2, we have $\lambda_1(D(G)) + \lambda_2(D(G)) > 2n - 4$ and $\lambda_k(D(G)) \geq -2$ for $k \geq 3$. It follows that

$$\begin{aligned}
S_k(D(G)) &= \lambda_1(D(G)) + \lambda_2(D(G)) + \lambda_3(D(G)) + \cdots + \lambda_k(D(G)) \\
&> 2n - 4 - 2(k-2) = 2n - 2k.
\end{aligned}$$

The proof is completed. □

From Theorem 2, we get Lin's result immediately.

Corollary 1 (*[3]*). *Let $k \geq 2$ be an integer and n a sufficiently large number with respect to k. Let G be a tree of order n. Then $S_k(D(G)) \geq 2n - 2k$ where the equality holds if and only if $G = K_{1,n-1}$.*

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