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Integral Cayley graphs over dihedral groups

Lu Lu[1] · **Qiongxiang Huang**[1] · **Xueyi Huang**[1]

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Abstract In this paper, we give a necessary and sufficient condition for the integrality of Cayley graphs over the dihedral group $D_n = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1}\rangle$. Moreover, we also obtain some simple sufficient conditions for the integrality of Cayley graphs over D_n in terms of the Boolean algebra of $\langle a\rangle$, from which we find infinite classes of integral Cayley graphs over D_n. In particular, we completely determine all integral Cayley graphs over the dihedral group D_p for a prime p.

Keywords Cayley graph · Integral graph · Dihedral group · Character

Mathematics Subject Classification 05C50

1 Introduction

A graph X is said to be *integral* if all eigenvalues of the adjacency matrix of X are integers. The property was first defined by Harary and Schwenk [12], who suggested the problem of classifying integral graphs. This problem initiated a significant investigation among algebraic graph theorists, trying to construct and classify integral graphs. Although this problem is easy to state, it turns out to be extremely hard. It has been attacked by many mathematicians during the past 40 years, and it is still wide open.

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✉ Qiongxiang Huang
huangqx@xju.edu.cn

1 College of Mathematics and Systems Science, Xinjiang University, Ürümqi 830046, Xinjiang, People's Republic of China

Since the general problem of classifying integral graphs seems too difficult, graph theorists started to investigate special classes of graphs, including trees, graphs with bounded degrees, regular graphs, and Cayley graphs. The first considerable result on integral trees was given by Watanabe and Schwenk in [21] and [22]. Then, many mathematicians constructed some infinite classes of integral trees with bounded diameters ([6,20,23]). It is heartening that Csikvári [9] constructed integral trees with arbitrarily large diameters. With regard to integral regular graphs, the first significant result was given by Bussemaker and Cvetković [7] in 1976, which showed that there are only 13 connected cubic integral graphs. About 20 years later, in 2000, Stevanović considered the 4-regular integral graphs avoiding ± 3 in the spectrum and gave the possible spectra of 4-regular bipartite integral graphs without ± 3 as their eigenvalues in [19]. Recently, Lepović [16] proposed that there exist exactly 93 non-regular, bipartite integral graphs with maximum degree 4.

Given a finite group G and a subset $1 \notin S \subseteq G$ with $S = S^{-1}$, the Cayley graph $X(G, S)$ has vertex set G and two vertices a, b are adjacent if $a^{-1}b \in S$. In 1982, Bridges and Mena [5] gave a complete characterization of integral graphs over abelian groups and So [18] found a new proof for integral circulant graphs in 2005. Later, Abdollahi and Vatandoost [1] showed that there are exactly seven connected cubic integral Cayley graphs in 2009. About the same year, Klotz and Sander [14] proved that, for an abelian group G, if the Cayley graph $X(G, S)$ is integral, then S belongs to the Boolean algebra $B(\mathcal{F}_G)$ generated by the subgroups of G. Moreover, they conjectured that the converse is also true, which has been proved by Alperin and Peterson [3].

In 2014, Cheng, Lau and Wong (cf. [8, Corollary 1.2]) presented that the normal Cayley graphs over symmetric groups are integral (a Cayley graph is said to be *normal* if its generating set S is closed under conjugation). In the same year, Alperin [2] gave a theoretical characterization of integral Cayley graphs over finite groups; however, it is far from being solved to obtain an explicitly characterization of integral Cayley graphs over non-abelian groups. As a simple attempt to this aspect, we try to characterize integral Cayley graphs over dihedral groups. At first, by using the expression of spectra of Cayley graphs, we obtain the necessary and sufficient conditions for the integrality of Cayley graphs over the dihedral group D_n (see Theorems 2.2 and 3.3). In terms of atoms of Boolean algebra of D_n, we also obtain a simple sufficient condition (see Corollary 3.2) and a necessary condition (see Corollary 3.3) for the integrality of Cayley graphs over D_n. In particular, we determine all integral Cayley graphs over D_p for a prime p (see Theorem 4.2).

2 The Spectra of Cayley Graphs Over Dihedral Groups

First of all, we review some basic definitions and notations of representation theory for latter use.

Let G be a finite group and V an n-dimensional vector space over $\mathbb{C}$. A *representation* of G on V is a group homomorphism $\rho : G \to GL(V)$, where $GL(V)$ denotes the group of automorphisms of V. The *degree* of ρ is the dimension of V. Two representations $\rho_1 : G \to GL(V_1)$ and $\rho_2 : G \to GL(V_2)$ of G are called *equivalent*, written as $\rho_1 \sim \rho_2$, if there exists an isomorphism $T : V_1 \to V_2$ such that $T\rho_1(g) = \rho_2(g)T$ for all $g \in G$.

Let $\rho : G \to GL(V)$ be a representation. The *character* $\chi_\rho : G \to \mathbb{C}$ of ρ is defined by setting $\chi_\rho(g) = Tr(\rho(g))$ for $g \in G$, where $Tr(\rho(g))$ is the trace of the representation matrix of $\rho(g)$ with respect to some basis of V. The *degree* of the character χ_ρ is just the degree of ρ, which equals $\chi_\rho(1)$. A subspace W of V is said to be G*-invariant* if $\rho(g)w \in W$ for each $g \in G$ and $w \in W$. If W is a G-invariant subspace of V, then the restriction of ρ on W, i.e., $\rho_{|W} : G \to GL(W)$, is a representation of G on W. Obviously, $\{1\}$ and V are always G-invariant subspaces, which are called *trivial*. We say that ρ is an *irreducible representation* and χ_ρ an *irreducible character* of G, if V has no non-trivial G-invariant subspaces. One can refer to [17] for more information about representation theory.

If we build synthetically a vector space $\mathbb{C}G$ whose basis consists of the elements of G, i.e.,

$$\mathbb{C}G = \left\{ \sum_{g\in G} c_g g \mid c_g \in \mathbb{C} \right\},$$

then the *(left) regular representation* of G is the homomorphism $L : G \to GL(\mathbb{C}G)$ defined by

$$L(g) \sum_{h\in G} c_h h = \sum_{h\in G} c_h gh = \sum_{x\in G} c_{g^{-1}x} x$$

for each $g \in G$. The following result about regular representation is well known.

Lemma 2.1 ([17]) *Let L be the regular representation of G. Then*

$$L \sim d_1\rho_1 \oplus d_2\rho_2 \oplus \cdots \oplus d_h\rho_h,$$

where $\rho_1, \dots, \rho_h$ are all non-equivalent irreducible representations of G and d_i is the degree of ρ_i $(1 \le i \le h)$.

Suppose that $X = X(G, S)$ is an undirected Cayley graph without loops, that is, S is inverse-closed and does not contain the identity. Let L be the regular representation of G, and $R(g)$ the representation matrix corresponding to $L(g)$ for $g \in G$. Babai [4] noted the adjacency matrix of $X(G, S)$ can be expressed in terms of $R(g)$.

Lemma 2.2 ([4]) *Let G be a finite group of order n, and let $S \subseteq G\backslash\{1\}$ be such that $S = S^{-1}$. Then, the adjacency matrix A of $X(G, S)$ can be expressed as $A = \sum_{s\in S} R(s)$, where $R(s)$ is the representation matrix corresponding to $L(s)$.*

Let $\rho_1, \dots, \rho_h$ be all non-equivalent irreducible representations of G with degrees $d_1, \dots, d_h$ $(d_1^2 + \cdots + d_h^2 = n)$, respectively. Denote by $R_i(g)$ the representation matrix corresponding to $\rho_i(g)$ for $g \in G$ and $1 \le i \le h$. Therefore, by Lemma 2.1,

$$L \sim d_1\rho_1 \oplus \cdots \oplus d_h\rho_h,$$

Table 1 Character table of D_n for odd n

	a^k	ba^k
ψ_1	1	1
ψ_2	1	-1
χ_h	$2\cos\left(\frac{2kh\pi}{n}\right)$	0

which means that there exists an orthogonal matrix P such that

$$PR(g)P^{-1} = d_1R_1(g) \oplus d_2R_2(g) \oplus \cdots \oplus d_hR_h(g)$$

for each $g \in G$. Therefore, we have

$$PAP^{-1} = P\left(\sum_{s\in S} R(s)\right)P^{-1} = d_1\sum_{s\in S} R_1(s) \oplus \cdots \oplus d_h\sum_{s\in S} R_h(s).$$

Suppose that $\lambda_{i,1}, \ldots, \lambda_{i,d_i}$ are all eigenvalues of the matrix $\sum_{s\in S} R_i(s)$ $(1 \le i \le h)$. The spectrum of $X(G, S)$ is given by

$$\mathrm{Spec}(X(G,S)) = \left\{[\lambda_{1,1}]^{d_1}, \ldots, [\lambda_{1,d_1}]^{d_1}, \ldots, [\lambda_{h,1}]^{d_h}, \ldots, [\lambda_{h,d_h}]^{d_h}\right\}.$$

Babai thus proved

Lemma 2.3 ([4]) *The spectrum of $X(G, S)$ is given by*

$$\mathrm{Spec}(X(G,S)) = \left\{[\lambda_{1,1}]^{d_1}, \ldots, [\lambda_{1,d_1}]^{d_1}, \ldots, [\lambda_{h,1}]^{d_h}, \ldots, [\lambda_{h,d_h}]^{d_h}\right\},$$

where $\lambda_{i,1}^t + \lambda_{i,2}^t + \cdots + \lambda_{i,d_i}^t = \sum_{s_1,\ldots,s_t\in S} \chi_{\rho_i}\left(\prod_{k=1}^t s_k\right)$ for any natural number t.

Denote by $D_n = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1}\rangle$ the dihedral group of order $2n$. Now we list the character table of D_n.

Lemma 2.4 ([17]) *The character table of D_n is given in Table 1 if n is odd, and in Table 2 otherwise, where ψ_i and χ_j are irreducible characters of degree one and two, respectively, and $1 \le h \le \lfloor\frac{n-1}{2}\rfloor$.*

Lemmas 2.3 and 2.4 yield the spectra of Cayley graphs over D_n immediately, as in [4, Proposition 4.1].

Theorem 2.1 ([4]) *Let D_n be a dihedral group and $S \subseteq D_n\backslash\{1\}$ satisfying $S = S^{-1}$. Then*

$$\mathrm{Spec}(X(D_n,S)) = \left\{[\lambda_i]^1; [\mu_{h1}]^2, [\mu_{h2}]^2 \mid i = 1, \ldots, m; h = 1, 2, \ldots, \lfloor\frac{n-1}{2}\rfloor\right\},$$

Table 2 Character table of D_n for even n

	a^k	ba^k
ψ_1	1	1
ψ_2	1	-1
ψ_3	$(-1)^k$	$(-1)^k$
ψ_4	$(-1)^k$	$(-1)^{k+1}$
χ_h	$2\cos\left(\frac{2kh\pi}{n}\right)$	0

where $m = 2$ if n is odd and $m = 4$ otherwise, and

$$\begin{cases} \lambda_i = \sum_{s\in S} \psi_i(s), & i = 1,\dots,m; \\ \mu_{h1} + \mu_{h2} = \sum_{s\in S} \chi_h(s), & h = 1,2,\dots,\lfloor\frac{n-1}{2}\rfloor; \\ \mu_{h1}^2 + \mu_{h2}^2 = \sum_{s_1,s_2\in S} \chi_h(s_1s_2), & h = 1,2,\dots,\lfloor\frac{n-1}{2}\rfloor. \end{cases} \tag{1}$$

Let A, B be two subsets of a group G. For any character χ of G, we denote $\chi(A) = \sum_{a\in A}\chi(a)$ and $\chi(AB) = \sum_{a\in A,b\in B}\chi(ab)$. Particularly, $\chi(A^2) = \sum_{a_1,a_2\in A}\chi(a_1a_2)$.

Theorem 2.2 *Let $D_n = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1}\rangle$ be the dihedral group, and let $S = S_1 \cup S_2 \subseteq D_n\backslash\{1\}$ be such that $S = S^{-1}$, where $S_1 \subseteq \langle a\rangle$ and $S_2 \subseteq b\langle a\rangle$. Then, $X(D_n, S)$ is integral if and only if the following two conditions hold for $1 \le h \le \lfloor\frac{n-1}{2}\rfloor$:*

(i) *$\chi_h(S_1)$, $\chi_h(S_1^2) + \chi_h(S_2^2)$ are integers;*

(ii) *$\Delta_h(S) = 2\left[\chi_h(S_1^2) + \chi_h(S_2^2)\right] - \left[\chi_h(S_1)\right]^2$ is a square number.*

Proof Note that $S_1S_2 = \{s_1s_2 \mid s_1 \in S_1, s_2 \in S_2\} \subseteq b\langle a\rangle$ and $S_2S_1 \subseteq b\langle a\rangle$. By Lemma 2.4, we have $\chi_h(S_1S_2) = 0 = \chi_h(S_2S_1)$. Thus,

$$\chi_h(S) = \sum\nolimits_{s_1\in S_1}\chi_h(s_1) + \sum_{s_2\in S_2}\chi_h(s_2) = \chi_h(S_1),$$

$$\begin{aligned} \chi_h(S^2) &= \sum\nolimits_{s_1,s_2\in S}\chi_h(s_1s_2) \\ &= \chi_h(S_1^2) + \chi_h(S_1S_2) + \chi_h(S_2S_1) + \chi_h(S_2^2) \\ &= \chi_h(S_1^2) + \chi_h(S_2^2). \end{aligned}$$

Then, the spectrum of $X(D_n, S)$ presented in (1) can be rewritten as

$$\begin{cases} \lambda_i = \sum_{s\in S}\psi_i(s) = \psi_i(S), & i = 1,\dots,m; \\ \mu_{h1} + \mu_{h2} = \chi_h(S_1), & h = 1,2,\dots,\lfloor\frac{n-1}{2}\rfloor; \\ \mu_{h1}^2 + \mu_{h2}^2 = \chi_h(S_1^2) + \chi_h(S_2^2), & h = 1,2,\dots,\lfloor\frac{n-1}{2}\rfloor. \end{cases} \tag{2}$$

First suppose that $X(D_n, S)$ is integral. From (2), we know that $\chi_h(S_1)$ and $\chi_h(S_1^2)+\chi_h(S_2^2)$ must be integers, and thus (i) holds. Since μ_{h1} and μ_{h2} are integers, and they are also the roots of the following quadratic equation:

$$x^2 - \chi_h(S_1)\cdot x + \frac{1}{2}\Big[\chi_h(S_1)^2 - (\chi_h(S_1^2) + \chi_h(S_2^2)\Big] = 0, \tag{3}$$

the discriminant $\Delta_h(S) = 2\left[\chi_h(S_1^2) + \chi_h(S_2^2)\right] - \left[\chi_h(S_1)\right]^2$ must be a square number, and thus (ii) follows.

Next suppose that (i) and (ii) hold. Then, the solutions μ_{h1} and μ_{h2} of (3) must be rational. This implies that μ_{h1} and μ_{h2} must be integers because they are algebraic integers. Additionally, the eigenvalues λ_i are always integers. Hence, $X(D_n, S)$ is integral. □

Let $C_n = \langle a\rangle$ be the cyclic group of order n. It is well known in [17] that the irreducible characters of C_n can be presented by

$$\phi_h(a^k) = e^{\frac{2hk\pi}{n}\mathbf{i}}, \text{ where } 0 \le h \le n-1. \tag{4}$$

Particularly, $\phi_0(a^k) = 1$. We have

$$\frac{1}{n}\sum_{k=0}^{n-1}\phi_h(a^k) = \langle\phi_h, \phi_0\rangle = 0 \text{ for } 1 \le h \le n-1. \tag{5}$$

Theoretically, Theorem 2.2 gives a necessary and sufficient condition for the integrality of Cayley graphs over dihedral groups. As an application of Theorem 2.2, we give a class of integral $X(D_n, S)$.

Corollary 2.1 *For odd number m, let $D_{2m} = \langle a, b \mid a^{2m} = b^2 = 1, bab = a^{-1}\rangle$ be the dihedral group of order $4m$. Let $S_1 = \{a^m\}$, $S_2 = b\langle a^2\rangle$ and $S = S_1 \cup S_2$. Then, $X(D_{2m}, S)$ is connected and integral.*

Proof It is easy to see that $S = S^{-1}$ generates D_{2m}, and so $X(D_{2m}, S)$ is connected. By Lemma 2.4, we have $\chi_h(S_1) = 2\cos(\frac{2hm\pi}{2m}) = \pm 2$, and $\chi_h(S_1^2) = 2$ due to $S_1^2 = \{1\}$. By simple calculation, S_2^2 consists of all elements of $\langle a^2\rangle$ in which each one appears m times, that is, $S_2^2 = m * \langle a^2\rangle$. From (4), (5) and Lemma 2.4, we have

$$\begin{aligned}
\chi_h(S_2^2) &= m\sum_{k=0}^{m-1}\chi_h(a^{2k}) = m\sum_{k=0}^{m-1}2\cos\frac{2h\cdot 2k\pi}{2m} = m\sum_{k=0}^{m-1}2\cos\frac{2hk\pi}{m}\\
&= m\sum_{k=0}^{m-1}\left(\cos\frac{2hk\pi}{m} + \cos\frac{2h(m-k)\pi}{m}\right)\\
&= m\left(\sum_{k=0}^{m-1}e^{\frac{2hk\pi}{m}\mathbf{i}} + \sum_{k=0}^{m-1}e^{\frac{2h(m-k)\pi}{m}\mathbf{i}}\right) = 2m\sum_{k=0}^{m-1}e^{\frac{2hk\pi}{m}\mathbf{i}}
\end{aligned}$$

$$= 2m \sum_{k=0}^{m-1} \phi_h((a^2)^k) = 2m \cdot m\langle \phi_h, \phi_0 \rangle = 0.$$

By Theorem 2.2, $X(D_{2m}, S)$ is integral. □

Remark 1 In fact, the spectrum of the graph $X(D_{2m}, S)$ in Corollary 2.1 can be obtained easily by Theorem 2.1, which is $\{-m-1, -m+1, m-1, m+1, [-1]^{2m-2}, [1]^{2m-2}\}$.

In the next section, we will simplify the result of Theorem 2.2 and provide infinite classes of integral Cayley graphs over dihedral groups in terms of Boolean algebra on cyclic groups.

3 The Necessary and Sufficient Condition for the Integrality of $X(D_n, S)$

Alperin and Peterson [3] give a necessary and sufficient condition for the integrality of Cayley graphs over abelian groups, in which they introduce some definitions such as Boolean algebra and atoms for a group. Let G be a finite group, and $\mathcal{F}_G$ the set consisting of all subgroups of G. Then, the *Boolean algebra* $B(G)$ is the set whose elements are obtained by arbitrarily finite intersections, unions, and complements of the elements in $\mathcal{F}_G$. The minimal non-empty elements of $B(G)$ are called *atoms*. Clearly, distinct atoms are disjoint. Alperin and Peterson show that each element of $B(G)$ is the union of some atoms, and each atom of $B(G)$ has the form $[g] = \{x \mid \langle x\rangle = \langle g\rangle, x \in G\}$, where $g \in G$.

We say that a subset $S \subseteq G$ is *integral* if $\chi(S) = \sum_{s\in S} \chi(s)$ is an integer for every character χ of G. From Lemma 2.3, we know that S must be an integral set if the Cayley graph $X(G, S)$ is integral. The following elegant result gives a simple characterization of integral Cayley graphs over an abelian group G by using integral sets and atoms of $B(G)$.

Theorem 3.1 ([3], Theorem 5.1 and Corollary 7.2) *Let G be an abelian group. Then, $S \subseteq G$ is integral iff $S \in B(G)$ iff S is a union of atoms of $B(G)$ iff $X(G, S)$ is integral.*

However, the statement of Theorem 3.1 is not true for non-abelian groups and is not true for dihedral groups. Notice that the dihedral group D_n is the semidirect product of cyclic group C_n by C_2, i.e., $D_n = C_n \rtimes C_2$. In what follows, we characterize integral $X(D_n, S)$ by using the Boolean algebra of the cyclic group C_n.

Let S be a subset of G. A *multi-set* based on S, denoted by S^m, is defined by a multiplicity function $m_S : S \to \mathbb{N}$, where $m_S(s)$ ($s \in S$) counts how many times s appears in the multi-set. We set $m_S(s) = 0$ for $s \notin S$. The multi-set S^m is called *inverse-closed* if $m_S(s) = m_S(s^{-1})$ for each $s \in S$ and *integral* if $\chi(S^m) = \sum_{s\in S} m_S(s)\chi(s)$ is an integer for each character χ of G. Besides, if no confusion occurs, we might write $\chi(S^m) = \sum_{s\in S^m} \chi(s)$ instead of $\chi(S^m) = \sum_{s\in S} m_S(s)\chi(s)$.

For $S \in B(G)$, we know that S is the union of some atoms, say $S = [g_1]\cup[g_2]\cup\cdots\cup[g_k]$. Denote by $S^{m_{g_1,g_2,\ldots,g_k}}$ the multi-set with multiplicity function $m_{g_1,g_2,\ldots,g_k}$, where

$m_{g_1,g_2,\dots,g_k}(s) = m_i \in \mathbb{N}$ for each $s \in [g_i]$ and $1 \le i \le k$; that is, $S^{m_{g_1,g_2,\dots,g_k}} = m_1 * [g_1] \cup m_2 * [g_2] \cup \cdots \cup m_k * [g_k]$. We define $C(G) = \{S^{m_{g_1,g_2,\dots,g_k}} \mid S = [g_1] \cup [g_2] \cup \cdots \cup [g_k] \in B(G), g_i \in G, k \in \mathbb{N}\}$ to be the collection of all multi-sets such as $S^{m_{g_1,g_2,\dots,g_k}}$, which is called the *integral cone* over $B(G)$. By Theorem 3.1, T is an integral set of the abelian group G iff $T \in B(G)$. With respect to multi-sets, we get the similar result by the same method of [3, Theorem 5.1]. For a multi-subset T^m of G, let v_{T^m} be the vector indexed by the elements of G such that $v_{T^m}(a) = m_T(a)$ for any $a \in G$. If T^m is integral, as in the proof of [3, Theorem 5.1], we also get that $v_{T^m} = m_1 v_{[a_1]} + m_2 v_{[a_2]} + \cdots + m_k v_{[a_k]}$ and hence $T^m = m_1 * [a_1] \cup m_2 * [a_2] \cup \cdots \cup m_k * [a_k] \in C(G)$. It follows that

Lemma 3.1 *Let G be an abelian group, and T^m a multi-subset of G. Then T^m is integral if and only if $T^m \in C(G)$, where $C(G)$ is the integral cone over $B(G)$.*

Remark 2 This result was first given by Bridges and Mena [5] in a different way and generalized by DeVos et al. [11].

Lemma 3.2 *Let U be a multi-set of integers satisfying $U = -U$, and let n, h be two positive integers. Then $\sum_{u \in U} \cos \frac{2hu\pi}{n} = \sum_{u \in U} e^{\frac{2hu\pi}{n}\mathbf{i}}$.*

Proof Since $U = -U$, we have $\sum_{u \in U} e^{\frac{2hu\pi}{n}\mathbf{i}} = \sum_{u \in U} e^{\frac{2h(-u)\pi}{n}\mathbf{i}}$. Therefore,

$$
\begin{aligned}
2\sum_{u\in U} e^{\frac{2hu\pi}{n}\mathbf{i}} &= \sum_{u\in U} e^{\frac{2hu\pi}{n}\mathbf{i}} + \sum_{u\in U} e^{\frac{2h(-u)\pi}{n}\mathbf{i}} = \sum_{u\in U}\left(e^{\frac{2hu\pi}{n}\mathbf{i}} + e^{\frac{2h(-u)\pi}{n}\mathbf{i}}\right) \\
&= \sum_{u\in U} 2\cos\frac{2hu\pi}{n}.
\end{aligned}
$$

This completes the proof. □

Lemma 3.3 *Let $D_n = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1}\rangle$ be the dihedral group, and T^m an inverse-closed multi-subset of $\langle a\rangle$. Let χ_h and ϕ_h be the irreducible characters of D_n and $\langle a\rangle$, respectively. We have*

(1) $\chi_h(T^m) = 2\phi_h(T^m)$ *for* $1 \le h \le \lfloor\frac{n-1}{2}\rfloor$*;*
(2) $\chi_h(T^m)$ *is and integer for all* $1 \le h \le \lfloor\frac{n-1}{2}\rfloor$ *iff* $\phi_h(T^m)$ *is an integer for all* $0 \le h \le n-1$.

Proof Since T^m is inverse-closed, there exists a multi-set $U = -U$ of integers such that $T^m = \{a^u \mid u \in U\}$. By Lemma 3.2, we have

$$
\chi_h(T^m) = \sum_{u\in U}\chi_h(a^u) = \sum_{u\in U} 2\cos\frac{2hu\pi}{n} = 2\sum_{u\in U} e^{\frac{2hu\pi}{n}\mathbf{i}} = 2\sum_{u\in U}\phi_h(a^u) = 2\phi_h(T^m).
$$

Thus (1) holds.

Notice that both $\chi_h(T^m)$ and $\phi_h(T^m)$ are algebraic integers. We claim that $\chi_h(T^m)$ is an integer iff $\phi_h(T^m)$ is an integer for each $1 \le h \le \lfloor\frac{n-1}{2}\rfloor$. Next we consider $\phi_{n-h}(T^m)$ for $1 \le h \le \lfloor\frac{n-1}{2}\rfloor$. By Lemma 3.2, we have

$$\phi_{n-h}(T^m) = \sum_{u\in U} e^{\frac{2(n-h)u\pi}{n}\mathbf{i}} = \sum_{u\in U} \cos\frac{2(n-h)u\pi}{n} = \sum_{u\in U} \cos\frac{2hu\pi}{n}$$
$$= \sum_{u\in U} e^{\frac{2hu\pi}{n}\mathbf{i}} = \phi_h(T^m).$$

Thus $\phi_{n-h}(T^m)$ is an integer iff $\phi_h(T^m)$ is an integer. Also note that $\phi_0(T^m)$ and $\phi_{\frac{n}{2}}(T^m)$ (for even n) are always integers. It follows (2). □

By Lemmas 3.1 and 3.3, we have the following result.

Theorem 3.2 *Let $D_n = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1}\rangle$ be the dihedral group, and T^m an inverse-closed multi-set with $T \subseteq \langle a\rangle \subseteq D_n$. Then $\chi_h(T^m)$ is an integer for all $1 \le h \le \lfloor\frac{n-1}{2}\rfloor$ if and only if $T^m \in C(\langle a\rangle)$. In particular, $\chi_h(T)$ is an integer for all $1 \le h \le \lfloor\frac{n-1}{2}\rfloor$ if and only if $T \in B(\langle a\rangle)$.*

Recall that the atom of $B(G)$ containing $g \in G$ has the form $[g] = \{x \mid \langle x\rangle = \langle g\rangle, x \in G\}$. Thus, for cyclic group $\langle a\rangle$ of order n, the atom of $B(\langle a\rangle)$ containing $a^d \in \langle a\rangle$, where $d|n$, can be presented as $[a^d] = \{a^l \mid (l, n) = d\}$, where (l, n) stands for the greatest common divisor of l and n.

Lemma 3.4 *Let $D_n = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1}\rangle$ be the dihedral group and $T \subseteq \langle a\rangle$. If $T \in B(\langle a\rangle)$, then $2\chi_h(T^2) = (\chi_h(T))^2$ for $1 \le h \le \lfloor\frac{n-1}{2}\rfloor$.*

Proof Without loss of generality, suppose that $T = [a^{d_1}] \cup [a^{d_2}] \cup \cdots \cup [a^{d_k}] \subseteq \langle a\rangle$ with $d_i|n$ for $i = 1, \ldots, k$. Let $\Phi_i = \{1 \le l_i \le n \mid (l_i, n) = d_i\}$. We have $[a^{d_i}] = \{a^{l_i} \mid (l_i, n) = d_i\} = \{a^{l_i} \mid l_i \in \Phi_i\}$. Thus, by setting $\Phi = \cup_{i=1}^{i=k}\Phi_i$, we have $T = \{a^l \mid l \in \Phi\}$ and $T^2 = \{a^{s+t} \mid s, t \in \Phi\}$. By Lemma 3.2, we have

$$2\chi_h(T^2) = 2\sum_{s,t\in\Phi} \chi_h(a^{s+t}) = 2\sum_{s,t\in\Phi} 2\cos\frac{2h(s+t)\pi}{n} = 4\sum_{s,t\in\Phi} e^{\frac{2h(s+t)\pi}{n}\mathbf{i}}$$
$$= 4\left(\sum_{s\in\Phi} e^{\frac{2hs\pi}{n}\mathbf{i}}\right)\left(\sum_{t\in\Phi} e^{\frac{2ht\pi}{n}\mathbf{i}}\right) = 4\left(\sum_{s\in\Phi}\cos\frac{2hs\pi}{n}\right)\left(\sum_{t\in\Phi}\cos\frac{2ht\pi}{n}\right)$$
$$= \left(\sum_{s\in\Phi} 2\cos\frac{2hs\pi}{n}\right)\left(\sum_{t\in\Phi} 2\cos\frac{2ht\pi}{n}\right) = \left(\sum_{s\in\Phi}\chi_h(a^s)\right)\left(\sum_{t\in\Phi}\chi_h(a^t)\right)$$
$$= (\chi_h(T))^2.$$

This completes the proof. □

From Theorem 3.2 and Lemma 3.4, we get the following corollary immediately.

Corollary 3.1 *Let $D_n = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1}\rangle$ be the dihedral group and $T \subseteq \langle a\rangle$. If $T \in B(\langle a\rangle)$, then $\chi_h(T)$, $\chi_h(T^2)$ are integers and $2\chi_h(T^2)$ is a square number for all $1 \le h \le \lfloor\frac{n-1}{2}\rfloor$.*

Using these preparations above, the result of Theorem 2.2 can be simplified as the following theorem.

Theorem 3.3 *Let $D_n = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1}\rangle$ be the dihedral group of order $2n$, and let $S = S_1 \cup S_2 \subseteq D_n \backslash \{1\}$ be such that $S = S^{-1}$, where $S_1 \subseteq \langle a\rangle$ and $S_2 \subseteq b\langle a\rangle$. Then $X(D_n, S)$ is integral if and only if $S_1 \in B(\langle a\rangle)$ and $2\chi_h(S_2^2)$ is a square number for all $1 \le h \le \lfloor \frac{n-1}{2} \rfloor$.*

Proof First suppose that $X(D_n, S)$ is integral. By Theorem 2.2, $\chi_h(S_1)$ is an integer for all $1 \le h \le \lfloor \frac{n-1}{2} \rfloor$. By Theorem 3.2, $S_1 \in B(\langle a\rangle)$. Then, by Lemma 3.4, we have $2\chi_h(S_1^2) = \big(\chi_h(S_1)\big)^2$. Therefore,

$$\Delta_h(S) = 2\Big[\chi_h(S_1^2) + \chi_h(S_2^2)\Big] - \big[\chi_h(S_1)\big]^2 = 2\chi_h(S_2^2).$$

Again by Theorem 2.2, $\Delta_h(S) = 2\chi_h(S_2^2)$ is a square number for all $1 \le h \le \lfloor \frac{n-1}{2} \rfloor$.

Conversely, suppose that $S_1 \in B(\langle a\rangle)$ and $2\chi_h(S_2^2)$ is a square number for all $1 \le h \le \lfloor \frac{n-1}{2} \rfloor$. By Corollary 3.1, both $\chi_h(S_1)$ and $\chi_h(S_1^2)$ are integers for each h. Moreover, $\chi_h(S_2^2)$ must be an integer because $2\chi_h(S_2^2)$ is a square number and $\chi_h(S_2^2)$ is an algebraic integer. Therefore, $\chi_h(S_1^2) + \chi_h(S_2^2)$ is an integer. Since $S_1 \in B(\langle a\rangle)$ and $2\chi_h(S_2^2)$ is a square number, by Lemma 3.4, $\Delta_h(S) = 2\chi_h(S_2^2)$ is a square number. Thus, $X(D_n, S)$ is integral by Theorem 2.2. □

Theorem 3.3 gives a criterion to find integral $X(D_n, S)$, from which we get infinite classes of integral Cayley graphs over dihedral groups in the following corollary.

Corollary 3.2 *Let $D_n = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1}\rangle$ be the dihedral group of order $2n$, and let $S = S_1 \cup S_2 \subseteq D_n \backslash \{1\}$ be such that $S = S^{-1}$, where $S_1 \subseteq \langle a\rangle$ and $S_2 \subseteq b\langle a\rangle$. If $S_1, bS_2 \in B(\langle a\rangle)$, then $X(D_n, S)$ is integral.*

Proof By Theorem 3.3, it suffices to show that $2\chi_h(S_2^2)$ is a square number for each $1 \le h \le \lfloor \frac{n-1}{2} \rfloor$. Since $bS_2 \in B(\langle a\rangle)$, bS_2 and S_2 can be written as

$$bS_2 = [a^{d_1}] \cup [a^{d_2}] \cup \cdots \cup [a^{d_k}] \text{ and } S_2 = b[a^{d_1}] \cup b[a^{d_2}] \cup \cdots \cup b[a^{d_k}],$$

for some $d_i | n$ where $i = 1, \ldots, k$. Therefore,

$$(bS_2)^2 = \{a^{l_1 + l_2} \mid l_1, l_2 \in \Phi\} \text{ and } S_2^2 = \{a^{l_1 - l_2} \mid l_1, l_2 \in \Phi\},$$

where $\Phi = \{\ell \mid (\ell, n) \in \{d_1, \ldots, d_k\}\}$. Note that $\Phi = -\Phi \pmod n$. We have $S_2^2 = (bS_2)^2$. Since $bS_2 \in B(\langle a\rangle)$, by Lemma 3.4,

$$2\chi_h(S_2^2) = 2\chi_h\left((bS_2)^2\right) = \chi_h^2(bS_2)$$

is a square number. □

Corollary 3.2 gives an explicit condition for the integrality of $X(D_n, S)$. However, this sufficient condition is not necessary. We will give a counterexample in Example 1. For this purpose, we need to introduce the famous Ramanujan sum. Let $s \ge 0, n \ge 1$

be two integers. The *Ramanujan sum* is defined by $c(s,n)=\sum_{(k,n)=1} e^{\frac{2sk\pi}{n}\mathbf{i}}$, which is known to be an integer (see [10] for reference):

$$c(s,n)=\frac{\varphi(n)}{\varphi\left(\frac{n}{(s,n)}\right)}\mu\left(\frac{n}{(s,n)}\right) \tag{6}$$

where $\varphi(\cdot)$ and $\mu(\cdot)$ are the Euler's totient function and Möbius function respectively. Therefore, it is easy to see that $c(s,n)=\mu(n)$ if $(s,n)=1$ and $c(s,n)=\varphi(n)$ if $(s,n)=n$.

Example 1 Let $D_{3k}=\langle a,b \mid a^{3k}=b^2=1, bab=a^{-1}\rangle$ be the dihedral group, and $S=S_1\cup S_2$, where $S_1=[a]$ and $S_2=\{b,ba^k\}$. Clearly, $X(D_{3k},S)$ is connected. By simple computation, $S_2^2=2*[1]\cup[a^k]$ is a multi-subset with multiplicity function: $m_{S_2^2}(x)=2$ if $x\in[1]=[a^n]=\{1\}$, and $m_{S_2^2}(x)=1$ if $x\in[a^k]=\{a^k,a^{2k}\}$. Therefore, by Lemma 3.1, $S_2^2\in C(\langle a\rangle)$ is integral. Moreover, by Lemma 3.3 (1), we have $2\chi_h(S_2^2)=4\phi_h(S_2^2)$. Therefore,

$$\begin{aligned}2\chi_h(S_2^2)=4\phi_h(S_2^2)&=4\left(2e^{\frac{2hn\pi}{n}\mathbf{i}}+\sum_{(l,n)=k}e^{\frac{2hl\pi}{n}\mathbf{i}}\right)=4\left(2+\sum_{(s,3)=1}e^{\frac{2hs\pi}{3}\mathbf{i}}\right)\\&=4\left(2+c(h,3)\right).\end{aligned}$$

If $3\nmid h$, then $(h,3)=1$ and so $c(h,3)=\mu(3)=-1$. It means that $2\chi_h(S_2^2)=4(2-1)=4$. If $3|h$, then $(h,3)=3$ and so $c(h,3)=\varphi(3)=2$. It means that $2\chi_h(S_2^2)=4(2+2)=16$. Thus $2\chi_h(S_2^2)$ is a square number for each $1\le h\le\lfloor\frac{3k}{2}\rfloor$. Moreover, $S_1=[a]\in B(\langle a\rangle)$. By Theorem 3.3, $X(D_{3k},S)$ is integral. However, $bS_2=\{1,a^k\}\notin B(\langle a\rangle)$.

Next we present a necessary condition for the integrality of $X(D_n,S)$.

Corollary 3.3 *Let $D_n=\langle a,b\mid a^n=b^2=1, bab=a^{-1}\rangle$ be the dihedral group, and let $S=S_1\cup S_2\subseteq D_n\backslash\{1\}$ be such that $S=S^{-1}$, where $S_1\subseteq\langle a\rangle$ and $S_2\subseteq b\langle a\rangle$. If $X(D_n,S)$ is integral, then $S_1\in B(\langle a\rangle)$ and $S_2^2\in C(\langle a\rangle)$.*

Proof If $X(D_n,S)$ is integral, then $S_1\in B(\langle a\rangle)$ and $2\chi_h(S_2^2)$ is a square number for all $1\le h\le\lfloor\frac{n-1}{2}\rfloor$ by Theorem 3.3, and so $\chi_h(S_2^2)$ must be a rational number. Thus, we claim that $\chi_h(S_2^2)\in\mathbb{Z}$ because $\chi_h(S_2^2)$ is an algebraic integer. By Theorem 3.2, we get $S_2^2\in C(\langle a\rangle)$. □

Unfortunately, the necessary condition given in Corollary 3.3 is not sufficient yet. We present a counterexample below.

Example 2 Let $D_7=\langle a,b\mid a^7=b^2=1, bab=a^{-1}\rangle$ be the dihedral group of order 14 and $S=S_2=\{ba,ba^2,ba^4\}$. It is clear that $X(D_7,S)$ is connected. By simple computation, we have

$$S^2=\{1,1,1,a,a^2,a^3,a^4,a^5,a^6\}=3*[1]\cup[a]\in C(\langle a\rangle).$$

Therefore, by Lemma 3.1, $S_2^2 \in C(\langle a\rangle)$ is integral. Moreover, by Lemma 3.3 (1), we have $2\chi_h(S_2^2) = 4\phi_h(S_2^2)$. Therefore, $2\chi_h(S_2^2) = 4\phi_h(S_2^2) = 4\left(3 + \sum_{k=1}^{6} e^{\frac{2hk\pi}{7}\mathbf{i}}\right) = 4(3 + c(h, 7))$. If $7 \nmid h$, then $(h, 7) = 1$ and so $c(h, 7) = \mu(7) = -1$. It means that $2\chi_h(S_2^2) = 4(3 - 1) = 8$, which is not a square number. By Theorem 3.3, $X(D_7, S)$ is not integral.

Although the integral Cayley graphs over dihedral groups are completely characterized by Theorem 3.3, it seems difficult to explicitly give all integral $X(D_n, S)$. In the next section, we will determine all integral $X(D_n, S)$ for n being a prime.

4 Integral Cayley Graphs Over D_p

Let $D_n = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1}\rangle$ be the dihedral group of order $2n$, and let $S = S_1 \cup S_2 \subseteq D_n \backslash \{1\}$ be such that $S = S^{-1}$, where $S_1 \subseteq \langle a\rangle$ and $S_2 \subseteq b\langle a\rangle$. We have known that $S_1 \in B(\langle a\rangle)$ and $S_2^2 \in C(\langle a\rangle)$ if $X(D_n, S)$ is integral by Corollary 3.3. This implies that S_2^2 has the form

$$S_2^2 = m_1 * [a^{d_1}] \cup m_2 * [a^{d_2}] \cup \cdots \cup m_k * [a^{d_k}], \tag{7}$$

for some $d_i | n$ where $i = 1, \ldots, k$. The multiplicity function $m_{S_2^2}$ of the multi-set S_2^2 is given by $m_{S_2^2}(x) = m_i$ for $x \in [a^{d_i}]$. We say that S_2^2 is *k-integral* if S_2^2 has the form of (7) with $m_i \neq 0$ for $1 \leq i \leq k$. Clearly, S_2^2 always contains 1, and if S_2 contains two distinct elements, then S_2^2 will contain an element different from 1. Taking $k = 1$ in (7), then S_2 contains only one element and so $S_2^2 = \{1\}$, thus $2\chi_h(S_2^2) = 4$ is a square number. Then, we have the following result for the 1-integral S_2^2.

Lemma 4.1 *Let $D_n = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1}\rangle$ and $S = S_1 \cup S_2 \subseteq D_n\backslash\{1\}$ such that $S = S^{-1}$, where $S_1 \subseteq \langle a\rangle$ and $S_2 \subseteq b\langle a\rangle$. If $S_1 \in B(\langle a\rangle)$ and $S_2 = \{ba^i\}$ for any $0 \leq i \leq n-1$, then $X(D_n, S)$ is integral.*

In what follows, we focus on 2-integral S_2^2, i.e., $S_2^2 = m_1 * [a^{d_1}] \cup m_2 * [a^{d_2}]$. Since $S_2 \subseteq b\langle a\rangle$, there exists $U \subset \mathbb{Z}_n$ such that $S_2 = \{ba^i \mid i \in U\}$. Then $S_2^2 = \{a^{u_1 - u_2} \mid u_1, u_2 \in U\}$ is a multi-set containing $t * \{1\}$, where $t = |U| = |S_2|$. Thus, without loss of generality, we always assume that $m_1 = t$ and $d_1 = n$. It is interesting to notice that there is a relation between the 2-integral sets and the difference sets in a group. Let G be a group of order n with a subset S of order k. Then S is called a (n, k, λ)-*difference set* in G provided that the differences ss'^{-1} for $s, s' \in S, s \neq s'$ contain every non-identity element of G exactly λ times. Difference sets are important in design theory because they are equivalent to symmetric (n, k, λ) designs with a regular automorphism group [15]. We refer the readers to [13] for more information about difference sets. By the arguments above, if S_2^2 is 2-integral, then $S_2^2 = t * [1] \cup m_2 * [a^{d_2}]$. If additional $[a^{d_2}] = \langle a\rangle\backslash\{1\}$, then $S_2' = bS_2$ is just the (n, t, m_2)-difference set in $\langle a\rangle$ because $S'S'^{-1} = S_2^2$. In fact, we will see that the difference sets in C_p for a prime p are very important when we consider the integral Cayley graphs in D_p. The underlying relation between difference sets in $\langle a\rangle$ and integral Cayley graphs over D_n is an interesting area of future research.

Suppose that $S_2 \in b\langle a\rangle$. By Lemma 3.3 (1), $\chi_h(S_2^2) = 2\phi_h(S_2^2)$ for $1 \le h \le \lfloor\frac{n-1}{2}\rfloor$. Thus $2\chi_h(S_2^2)$ is a square number if and only if $\phi_h(S_2^2)$ is a square number. The following result gives a necessary and sufficient condition for $\phi_h(S_2^2)$ being a square number when S_2^2 is 2-integral.

Lemma 4.2 *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ ($p_i \ge 3$, $r \ge 1$) be the prime factorization of n and $S_2 \subseteq b\langle a\rangle \subseteq D_n = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1}\rangle$. If $S_2^2 = t * [1] \cup m_2 * [a^{d_2}]$ (implying that $|S_2| = t$) is 2-integral, then $\phi_h(S_2^2)$ is a square number for all $1 \le h \le \lfloor\frac{n-1}{2}\rfloor$ if and only if $\frac{n}{d_2} = p_i$ for some i ($1 \le i \le r$), and $t = p_i - 1$ or p_i.*

Proof Let $n_2 = \frac{n}{d_2}$, according to (4) and (6) we have

$$\begin{aligned}
\phi_h(S_2^2) &= \sum_{x\in S_2^2} \phi_h(x) = t\phi_h(1) + \sum_{x\in[a^{d_2}]} m_2\phi_h(x)\\
&= t + m_2 \sum_{(l,n)=d_2} e^{\frac{2hl\pi}{n}\mathbf{i}} = t + m_2 \sum_{(l/d_2,n_2)=1} e^{\frac{2hl/d_2\pi}{n_2}\mathbf{i}}\\
&= t + m_2 c(h, n_2) = t + m_2 \frac{\varphi(n_2)}{\varphi\left(\frac{n_2}{(h,n_2)}\right)}\mu\left(\frac{n_2}{(h, n_2)}\right). \qquad (8)
\end{aligned}$$

By counting the number of elements of S_2^2, we have

$$t + m_2\varphi(n_2) = t^2. \qquad (9)$$

From (8) and (9) we have

$$\phi_h(S_2^2) = t + \frac{t(t-1)}{\varphi\left(\frac{n_2}{(h,n_2)}\right)}\mu\left(\frac{n_2}{(h, n_2)}\right). \qquad (10)$$

Now we consider the sufficiency. Suppose that there exists some i such that $n_2 = p_i$, and $t = p_i - 1$ or p_i. For any h satisfying $(h, n_2) = 1$, from (10) we have

$$\phi_h(S_2^2) = \frac{t(p_i - t)}{p_i - 1}. \qquad (11)$$

Thus $\phi_h(S_2^2) = 1$ if $t = p_i - 1$, and $\phi_h(S_2^2) = 0$ if $t = p_i$. Additionally, for any h satisfying $(h, n_2) = n_2$, from (10) we have $\phi_h(S_2^2) = t^2$. Thus, $\phi_h(S_2^2)$ is always a square number for $1 \le h \le \lfloor\frac{n-1}{2}\rfloor$.

Conversely, assume that $\phi_h(S_2^2)$ is a square number, say $\phi_h(S_2^2) = w_h^2$ ($w_h \ge 0$) for $1 \le h \le \lfloor\frac{n-1}{2}\rfloor$. Notice that n_2 is a factor of n. We need to consider the following three cases:

Case 1. $n_2 = p_i$ for some i ($1 \le i \le r$);

Taking h such that $(h, n_2) = 1$, then we also have (11), from which we obtain that $t > w_h^2$ due to $t \geq 2$, and

$$p_i = \frac{(t + w_h)(t - w_h)}{t - w_h^2}. \tag{12}$$

Note that $t - w_h \geq t - w_h^2 \geq 1$. First assume that $t - w_h = 1$. Then $t - w_h = t - w_h^2 = 1$, and so $w_h = 0$ or $w_h = 1$. If $w_h = 0$, then $t = w_h + 1 = 1$, which is impossible. If $w_h = 1$, then $t = 2$, and $p_i = 3$ by (12). Next assume that $t - w_h > 1$. Since p_i is a prime, from (12) we have $t - w_h = t - w_h^2$ or $t + w_h = t - w_h^2$. If $t + w_h = t - w_h^2$, then $w_h = 0$; if $t - w_h = t - w_h^2$, then $w_h = 0$ or $w_h = 1$. Thus, from (12) we have $t = p_i$ if $w_h = 0$, and $t = p_i - 1$ if $w_h = 1$.

Case 2. $p_i^2 | n_2$ for some i $(1 \leq i \leq r)$;

By taking h such that $\frac{n_2}{(h,n_2)} = n_2$, we have $\mu\left(\frac{n_2}{(h,n_2)}\right) = \mu(n_2) = 0$, which leads to that $\phi_h(S_2^2) = t$ by (10). Thus $t = |S_2|$ is a square number independent with h. By taking another h such that $\frac{n_2}{(h,n_2)} = p_i$, we have $\phi_h(S_2^2) = \frac{t(p_i - t)}{p_i - 1}$ by (10), which gives $t = p_i - 1$ or p_i by the arguments of Case 1. Finally, if $t = p_i$ then p_i is a square number, a contradiction; if $t = p_i - 1$, we have $t^2 = (p_i - 1)^2 < p_i(p_i - 1) \leq \varphi(n_2)$, which contradicts (9). Thus, in this case, $\phi_h(S_2^2)$ cannot be always a square number for $1 \leq h \leq \lfloor \frac{n-1}{2} \rfloor$.

Case 3. $p_i p_j | n_2$ for some $i \neq j$ $(1 \leq i, j \leq r)$.

By taking h such that $\frac{n_2}{(h,n_2)} = p_i$, we have $\phi_h(S_2^2) = \frac{t(p_i - t)}{p_i - 1}$ by (10), which gives $t = p_i - 1$ or p_i by the arguments of Case 1. Similarly, by taking h such that $\frac{n_2}{(h,n_2)} = p_j$, we have $t = p_j - 1$ or p_j. Note that $t = |S_2|$ is independent with h and $i \neq j$. We have $p_i - 1 = p_j$ or $p_i = p_j - 1$, which are all impossible because both of p_i and p_j are odd primes. Thus $\phi_h(S_2^2)$ cannot be always a square number for $1 \leq h \leq \lfloor \frac{n-1}{2} \rfloor$ in this case.

This completes the proof. □

By Lemma 4.2, we gives a specific characterization of integral $X(D_n, S)$ for S_2^2 being 2-integral.

Theorem 4.1 *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ where $p_i \geq 3$ is a prime. Let $1 \notin S = S_1 \cup S_2 \subseteq D_n = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1} \rangle$ with $S = S^{-1}$, $S_1 \subseteq \langle a \rangle$ and $S_2 \subseteq b\langle a \rangle$. If S_2^2 is a 2-integral, then the Cayley graph $X(D_n, S)$ is integral if and only if $S_1 \in B(\langle a \rangle)$ and $S_2 = ba^j \langle a^{\frac{n}{p_i}} \rangle \backslash \{ba^{k\frac{n}{p_i}+j}\}$ or $ba^j \langle a^{\frac{n}{p_i}} \rangle$, where $1 \leq i \leq r$, $0 \leq k \leq p_i - 1$ and $0 \leq j \leq \frac{n}{p_i} - 1$.*

Proof We first consider the sufficiency. If $S_2 = ba^j \langle a^{\frac{n}{p_i}} \rangle \backslash \{ba^{k\frac{n}{p_i}+j}\}$, we have

$$\begin{aligned} S_2^2 &= \left(ba^j \langle a^{\frac{n}{p_i}} \rangle \cdot ba^j \langle a^{\frac{n}{p_i}} \rangle \backslash 2 * \left(ba^j \langle a^{\frac{n}{p_i}} \rangle \cdot \{ba^{k\frac{n}{p_i}+j}\}\right)\right) \cup \left\{ba^{k\frac{n}{p_i}+j} \cdot ba^{k\frac{n}{p_i}+j}\right\} \\ &= \left(p_i * \langle a^{\frac{n}{p_i}} \rangle \backslash 2 * \langle a^{\frac{n}{p_i}} \rangle\right) \cup \{1\} \\ &= (p_i - 1) * \{1\} \cup (p_i - 2) * [a^{\frac{n}{p_i}}]. \end{aligned}$$

Similarly, if $S_2 = b\langle a^{\frac{n}{p_i}}\rangle a^j$, we have $S_2^2 = p_i * \{1\} \cup p_i * [a^{\frac{n}{p_i}}]$. Thus, by Lemma 4.2, $\phi_h(S_2^2)$ is a square number for all $1 \le h \le \lfloor\frac{n-1}{2}\rfloor$, so are $2\chi_h(S_2^2)(= 4\phi_h(S_2^2))$. By Theorem 3.3, $X(D_n, S)$ is integral and the sufficiency follows.

For the necessity, let S_2^2 be 2-integral, we may assume that

$$S_2^2 = t * [1] \cup m_2 * \left[a^{d_2}\right] \tag{13}$$

where $t = |S_2| \ge 2$ and $d_2|n$. Since $X(D_n, S)$ is integral, $\phi_h(S_2^2)$ presented in (10) is a square number for all $1 \le h \le \lfloor\frac{n-1}{2}\rfloor$. By Lemma 4.2, there exists $1 \le i \le r$ such that $n_2 = \frac{n}{d_2} = p_i$ and $t = p_i - 1$ or p_i. If $t = p_i - 1$, we have $m_2 = \frac{t^2-t}{\varphi(n_2)} = p_i - 2$ from (9). Thus, we may assume that $S_2 = \{ba^{u_1}, \dots, ba^{u_{p_i-1}}\}$, combining (13) we have

$$\begin{aligned}(p_i - 1) * \{1\} \cup (p_i - 2) * [a^{\frac{n}{p_i}}] = S_2^2 &= \{a^{u_s-u_t} \mid 1 \le s, t \le p_i - 1\}\\ &= (p_i - 1) * \{1\} \cup \{a^{u_s-u_t} \mid s \ne t\}.\end{aligned}$$

Therefore, $a^{u_s-u_t} \in [a^{\frac{n}{p_i}}]$ for any $s \ne t$. Since $a^{u_2-u_1}, \dots, a^{u_{p_i-1}-u_1}$ are different elements in $[a^{\frac{n}{p_i}}]$, there exists k_1 $(1 \le k_1 \le p_i - 1)$ such that

$$\{a^{u_2-u_1}, \dots, a^{u_{p_i-1}-u_1}\} = [a^{\frac{n}{p_i}}]\backslash\{a^{k_1\frac{n}{p_i}}\}. \tag{14}$$

Note that u_1 can be written as $u_1 = k_2\frac{n}{p_i} + j$, where $0 \le k_2 \le p_i - 1$ and $0 \le j \le \frac{n}{p_i} - 1$. Then $a^{u_1} = a^{k_2\frac{n}{p_i}+j}$ and from (14) we have

$$\begin{aligned}bS_2 &= \left\{a^{u_1}, a^{u_2}, \dots, a^{u_{p_i-1}}\right\} = \left\{a^{u_1}\right\} \cup \left\{a^{u_2-u_1}, \dots, a^{u_{p_i-1}-u_1}\right\} \cdot \left\{a^{u_1}\right\}\\ &= \{a^{u_1}\} \cup \left([a^{\frac{n}{p_i}}]\backslash\{a^{k_1\frac{n}{p_i}}\}\right) \cdot \{a^{u_1}\}\\ &= \{a^{k_2\frac{n}{p_i}+j}, a^{(k_2+1)\frac{n}{p_i}+j}, \dots, a^{(k_2+p_i-1)\frac{n}{p_i}+j}\}\backslash\{a^{(k_1+k_2)\frac{n}{p_i}+j}\}\\ &= \left\{a^j, a^{\frac{n}{p_i}+j}, \dots, a^{(p_i-1)\frac{n}{p_i}+j}\right\} \backslash \left\{a^{k\frac{n}{p_i}+j}\right\} \quad \text{where } 0 \le k \le p_i - 1\\ &= \langle a^{\frac{n}{p_i}}\rangle a^j \backslash \left\{a^{k\frac{n}{p_i}+j}\right\}.\end{aligned}$$

Thus $S_2 = ba^j\langle a^{\frac{n}{p_i}}\rangle\backslash\{ba^{k\frac{n}{p_i}+j}\}$. Similarly, if $t = p_i$, we have $m_2 = p_i$. Assume that $S_2 = \{ba^{u_1}, \dots, ba^{u_{p_i}}\}$, combining (13) we have

$$\begin{aligned}p_i * \{1\} \cup p_i * [a^{\frac{n}{p_i}}] = S_2^2 &= \{a^{u_s-u_t} \mid 1 \le s, t \le p_i - 1\}\\ &= p_i * \{1\} \cup \{a^{u_s-u_t} \mid s \ne t\}.\end{aligned}$$

So we have $\{a^{u_2-u_1}, \dots, a^{u_{p_i}-u_1}\} = [a^{\frac{n}{p_i}}]$. Similarly, a^{u_1} can be written as $a^{u_1} = a^{k_2\frac{n}{p_i}+j}$, then we have $bS_2 = a^j\langle a^{\frac{n}{p_i}}\rangle$ and so $S_2 = ba^j\langle a^{\frac{n}{p_i}}\rangle$.

This completes the proof. □

Theorem 4.2 *For an odd prime p, let $D_p = \langle a, b \mid a^p = b^2 = 1, bab = a^{-1}\rangle$ and $S = S_1 \cup S_2$ such that $S = S^{-1}$, where $S_1 \subseteq \langle a\rangle$, $S_2 \subseteq b\langle a\rangle$. Then, the Cayley graph $X(D_p, S)$ is integral if and only if $S_1 \in B(\langle a\rangle) = \{\emptyset, \{1\}, [a], \langle a\rangle\}$ and $S_2 = b\langle a\rangle \setminus \{ba^k\}$, $b\langle a\rangle$ or $\{ba^k\}$ where $0 \le k \le p-1$.*

Proof If $S_2 = \{ba^k\}$, then $X(D_p, S)$ is integral by Lemma 4.1. If $S_2 = b\langle a\rangle \setminus \{ba^k\}$, then $S_2^2 = (p-1) * \{1\} \cup (p-2) * [a]$, thus $d_2 = 1$ and $t = |S_2| = p - 1$. By Lemma 4.2, $\phi_h(S_2^2)$ is a square number for all $1 \le h \le \lfloor \frac{p-1}{2} \rfloor$, so are $2\chi_h(S_2^2)$. Therefore, by Theorem 3.3, $X(D_p, S)$ is integral. Similarly, if $S_2 = b\langle a\rangle$, we have $S_2^2 = p * \{1\} \cup p * [a]$, and so $X(D_p, S)$ is integral. We get the sufficiency. In what follows, we consider the necessity.

Suppose $X(D_p, S)$ is integral. By Corollary 3.3, we have $S_1 \in B(\langle a\rangle) = \{\emptyset, \{1\}, [a], \langle a\rangle\}$ and $S_2^2 \in C(\langle a\rangle)$. First suppose that $t = |S_2| = 1$. We have $S_2 = \{ba^k\}$ for some $0 \le k \le p-1$. Next suppose that $t = |S_2| > 1$. We see that S_2^2 is 2-integral because $B(\langle a\rangle)$ has only two atoms. By Theorem 4.1, we have $|S_2| = p-1$ or p. If $|S_2| = p-1$, we obtain that $S_2 = b\langle a\rangle \setminus \{ba^k\}$ where $0 \le k \le p-1$ (note that $S_2 = b[a]$ while $k = 0$); if $|S_2| = p$, we obtain that $S_2 = b\langle a\rangle = \{b\} \cup b[a]$. $\square$

Remark 3 Recall the definition of difference sets. By the proof of Theorem 4.2, we note that $S_2' = bS_2$ is a $(p, p-1, p-2)$-difference set and a (p, p, p)-difference set in $\langle a\rangle$ when $S_2 = b\langle a\rangle \setminus \{ba^k\}$ and $S_2 = b\langle a\rangle$ in Theorem 4.2, respectively.

By Corollary 3.3, if $X(D_n, S)$ is integral then $S_1 \in B(\langle a\rangle)$ and $S_2^2 \in C(\langle a\rangle)$, which means that S_1 and S_2^2 are clearly found. If we can obtain S_2 from (7), the integral $X(D_n, S)$ will be finally determined by verifying if $2\chi_h(S_2^2)$ is square number (see Theorem 3.3). However, it seems difficult to do this even if n is a prime. Example 2 provides an instance that $S_2^2 = \{1, 1, 1, a, a^2, a^3, a^4, a^5, a^6\} = 3*[1] \cup [a] \in C(\langle a\rangle)$, but $S_2 = \{ba, ba^2, ba^4\}$ is not of the forms stated in Theorem 4.2 (i.e., $S_2 = b\langle a\rangle \setminus ba^k$, $b\langle a\rangle$ or $\{ba^k\}$). Hence $X(D_7, S_2)$ is not integral by Theorem 4.2. In fact, we have many such instances, say, $S_2 = \{ba, ba^3, ba^4, ba^8\} \subset b\langle a\rangle$, where $|\langle a\rangle| = 13$, is not of the forms stated in Theorem 4.2, but $S_2^2 = 4 * [1] \cup [a]$. Also, $X(D_{13}, S_2)$ is not integral by Theorem 4.2. Note that in the former example, $S_2' = bS_2 = \{a, a^2, a^4\}$ is a $(7, 3, 1)$-difference set in C_7, which is known as the Paley difference set; in the latter example, $S_2' = bS_2 = \{a, a^3, a^4, a^8\}$ is a $(13, 4, 1)$-difference set in C_{13}.

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